東北數學雜誌

第拾四巻

THE

TÔHOKU MATHEMATICAL JOURNAL

Edited by

T. Hayashi

with the cooperation of

M. Fujiwara, *J. Ishiwara,*

T. Kubota, *S. Kakeya,* *T. Kojima.*

Vol. 14.

1918.

THE TÔHOKU IMPERIAL UNIVERSITY.

SENDAI, JAPAN.

Contents.

THE TÔHOKU MATHEMATICAL JOURNAL

A Determinantal Theorem and Clifford's Theorem on n Lines,

by

TSURUICHI HAYASHI and KWAN SHIBATA, Sendai.

1. Denoting the determinant

$$(1)\qquad \begin{vmatrix} xy & x & y & 1 \\ \alpha_1'\beta_1' & \alpha_1' & \beta_1' & 1 \\ \alpha_2\beta_2 & \alpha_2 & \beta_2 & 1 \\ \alpha_3\beta_3 & \alpha_3 & \beta_3 & 1 \end{vmatrix}$$

by

$$D(x, \alpha_1', \alpha_2, \alpha_3),$$

we will at first prove the following :

Theorem 1. *If*

$$(2)\quad D(x, \alpha_1', \alpha_2, \alpha_3)=0, \quad D(x, \alpha_2', \alpha_3, \alpha_1)=0, \quad D(x, \alpha_3', \alpha_1, \alpha_2)=0$$

then the three equations

$$(3)\quad D(\xi, \alpha_1, \alpha_2', \alpha_3')=0, \quad D(\xi, \alpha_2, \alpha_3', \alpha_1')=0, \quad D(\xi, \alpha_3, \alpha_1', \alpha_2')=0$$

are not independent.

That is, if equations (2) and the first two, for instance, of equations (3) be assumed, the last of equations (3) can be deduced therefrom.

An algebraical proof. The determinant (1) can be transformed into[1]

$$(\alpha_1'-\alpha_2)(x-\alpha_3)(\beta_2-y)(\beta_1'-\beta_3)-(\alpha_2-x)(\alpha_1'-\alpha_3)(\beta_1'-\beta_2)(y-\beta_3).$$

Hence under the assumption, we get the five equations

$$\frac{(\alpha_1'-\alpha_2)(x-\alpha_3)}{(\alpha_2-x)(\alpha_1'-\alpha_3)}=\frac{(\beta_1'-\beta_2)(y-\beta_3)}{(\beta_2-y)(\beta_1'-\beta_3)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots$$

(1) Burnside and Panton, Theory of Equations, vol. 2, 1904, p. 55.

$$\frac{(\alpha_1-\alpha_2')(\xi-\alpha_3')}{(\alpha_2'-\xi)(\alpha_1-\alpha_3')}=\frac{(\beta_1-\beta_2')(\eta-\beta_3')}{(\beta_2'-\eta)(\beta_1-\beta_3')},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots\cdots\cdots$$

Multiplying up these equations, and taking the reciprocals of the two sides of the resulting equation, we arrive at

$$\frac{(\alpha_3-\alpha_1')(\xi-\alpha_2')}{(\alpha_1'-\xi)(\alpha_3-\alpha_2')}=\frac{(\beta_3-\beta_1')(\eta-\beta_2')}{(\beta_1'-\eta)(\beta_3-\beta_2')}.$$

Hence the conclusion.

A geometrical proof. If we take the ten numbers

$$(4) \qquad \begin{cases} \alpha_1, \ \alpha_2, \ \alpha_3, \ \beta_1, \ \beta_2, \ \beta_3, \\ \alpha_1', \ \alpha_2', \ \alpha_3', \ \beta_1', \ \beta_2', \ \beta_3', \\ \qquad x, \ y, \ \xi, \ \eta \end{cases}$$

as the coordinates of ten points forming a range on a line, the equation

$$D(x, \alpha_1', \alpha_2, \alpha_3)=0$$

means that the two sets of four points $(\alpha_1', \alpha_2, \alpha_3, x)$ and $(\beta_1', \beta_2, \beta_3, y)$ are in homographic relation. Hence if we denote the ten points by

$$(5) \qquad \begin{cases} A_1, \ A_2, \ A_3, \ B_1, \ B_2, \ B_3, \\ A_1', \ A_2', \ A_3', \ B_1', \ B_2', \ B_3' \\ \qquad X, \ Y, \ \Xi, \ H, \end{cases}$$

we are to prove that if the three sets

$$(6) \qquad \begin{cases} (A_1', A_2, A_3), (B_1', B_2, B_3); \\ (A_2', A_3, A_1), (B_2', B_3, B_1); \\ (A_3', A_1, A_2), (B_3', B_1, B_2) \end{cases}$$

have common homographic points (X, Y), the other three sets

$$(7) \qquad \begin{cases} (A_1, A_2', A_3'), (B_1, B_2', B_3'); \\ (A_2, A_3', A_1'), (B_2, B_3', B_1'); \\ (A_3, A_1', A_2'), (B_3, B_1', B_2') \end{cases}$$

have also common homographic points (Ξ, H). From (6) and the first two of (7), we get the five equations, which can be promptly written down,

$$\frac{A_1'A_3}{A_2A_3}\cdot\frac{A_2X}{A_1'X}=\frac{B_1'B_2}{B_2B_3}\cdot\frac{B_2Y}{B_2'Y},$$

$$\cdots\cdots\cdots\cdots\qquad\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\qquad\cdots\cdots\cdots\cdots$$

$$\frac{A_1A_3'}{A_2'A_3'}\cdot\frac{A_2'\Xi}{A_1\Xi}=\frac{B_1B_3'}{B_2'B_3'}\cdot\frac{B_3'H}{B_2H},$$

$$\cdots\cdots\cdots\cdots\qquad\cdots\cdots\cdots\cdots,$$

whence the result follows at once.

N. B. The two symbols x and y can be easily eliminated from the three equations (2). The result of elimination must remain unaltered, when the symbols α, β with dashes are interchanged with those without dashes. But it is not easy to show it directly.

Theorem 2. *If*

$$D(\alpha,\beta,\varepsilon,\varphi)=0,\qquad D(\gamma,\delta,\varepsilon,\varphi)=0,$$
$$D(\alpha,\delta,\theta,\varphi)=0,\qquad D(\beta,\gamma,\theta,\varphi)=0,$$

then four equations

$$D(\alpha,\beta,\theta,\psi)=0,\qquad D(\gamma,\delta,\theta,\psi)=0,$$
$$D(\alpha,\delta,\varepsilon,\psi)=0,\qquad D(\beta,\gamma,\varepsilon,\psi)=0$$

are not independent.

This can be proved by a similar way.

2. Now the equation to the circle in Cartesian rectangular coordinates (x, y) passing through the three given points $P_1=(x_1, y_1)$, $P_2=(x_2, y_2)$, $P_3=(x_3, y_3)$ is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0,$$

that is,

$$\begin{vmatrix} x^2+y^2 & x+iy & x-iy & 1 \\ x_1^2+y_1^2 & x_1+iy_1 & x_1-iy_1 & 1 \\ x_2^2+y_2^2 & x_2+iy_2 & x_2-iy_2 & 1 \\ x_3^2+y_3^2 & x_3+iy_3 & x_3-iy_3 & 1 \end{vmatrix}=0.$$

Hence if we put as usual

$$x=r\cos\theta,\qquad y=r\sin\theta,$$

the equation becomes

$$\begin{vmatrix} r^2 & [r,\theta] & [r,-\theta] & 1 \\ r_1^2 & [r_1,\theta_1] & [r_1,-\theta_1] & 1 \\ r_2^2 & [r_2,\theta_2] & [r_2,-\theta_2] & 1 \\ r_3^2 & [r_3,\theta_3] & [r_3,-\theta_3] & 1 \end{vmatrix} = 0,$$

in which $[r,\theta]$ and $[r,-\theta]$ stand for $re^{\theta\sqrt{-1}}$ and $re^{-\theta\sqrt{-1}}$ respectively. This determinant is a special one of the determinant (1) in which $|x|=|y|$ and $|\alpha|=|\beta|$. Hence by Theorem 1, we get the following geometrical theorem :

Theorem 3. *If the three circles passing through the three points*

$$(P_1', P_2, P_3), \qquad (P_2', P_3, P_1), \qquad (P_3', P_1, P_2)$$

respectively pass through a common point, the three circles passing through the three points

$$(P_1, P_2', P_3'), \qquad (P_2, P_3', P_1'), \qquad (P_3, P_1', P_2')$$

respectively also pass through a common point.

Or otherwise stated : *If four points on a circle or line*(1) *be taken in sequence and if each successive pair be connected by a circle, the remaining intersections of successive pairs of circles are concyclic or collinear.* Still another form for this theorem is as follows : *If four circles be arranged in sequence, each two successive circles intersecting, and a circle pass through one point of each such pair of intersections, then the remaining intersections lie on another circle or a line*(2).

Changing the symbols and their positions, we can state this theorem in the following form which is of use to us in the latter part of this paper : *If any five sets of four points*

$$(M_1, M_2, K_3, K_4),$$
$$(M_1, M_3, K_2, K_4),$$
$$(M_1, M_4, K_2, K_3),$$
$$(M_2, M_3, K_1, K_4),$$
$$(M_2, M_4, K_1, K_3),$$

---

(1) When four points (r,θ), (r_1,θ_1), (r_2,θ_2) and (r_3,θ_3) lie on one and the same line, the relation (8) also subsists.

(2) The lucid use of words for these enunciations, and for some of the enunciations mentioned later on, we are indebted to Prof. Coolidge's elegant and elaborate work "Treatise on the circle and the sphere" 1916. The word "four" in these enunciations may be changed into "$2n$."

$$(M_3, M_4, K_1, K_2),$$

be concyclic, the remaining set is also concyclic.

Theorem 3 can be easily proved by using the method of inversion from the theorem: If a point be marked on each side-line of a triangle, the three circles each through a vertex and the adjacent marked points are concurrent.

From Theorem 2, we get the following

Theorem 4. *If the four circles passing through the three points*

$$(A, B, E), \qquad (C, D, E), \qquad (A, D, F), \qquad (B, C, F)$$

respectively pass through a common point, and if the three circles passing through the three points

$$(A, B, F), \qquad (C, D, F), \qquad (A, D, E)$$

respectively pass through a common point, then the circle passing through the three points

$$(B, C, E)$$

passes through the common point of the latter three.

This theorem is feebler in efficacy than that which can be gotten by inversion from

Theorem 5. *If four lines be given, whereof no two are parallel nor any three concurrent, the circumscribing circles of the triangles which they form three by three are concurrent at the point called the Miquel point of the four lines.*

3. In this Journal several geometrical theorems have been proved by using complex numbers(1). Most lately Mr. S. Ôue has proved very elegantly also by using complex numbers a theorem on the circles associated with n points on the circumference of a given circle. One of us has found the theorem proved more early by Prof. J. L. Coolidge, but will stand security for the independence of Mr. Ôue's investigation, the methods of attack being very different(2). Now we have here proved, also by using complex numbers, Theorem 3 which is a corner-stone in Prof. Coolidge's paper.

---

(1) Hayashi, Vol. 4, p. 71; Fujiwara, Vol. 4, p. 75; Kakeya, Vol. 6, p. 241.

(2) Ôue, Vol. 10, p. 225. Coolidge, *Some circles associated with concyclic points*, Annals of Math., 2nd ser., vol, 12, 1910–1911, p. 39. Also consult his work, Treatise on the circle and the sphere, 1916, p. 94, and Agronomof's paper, *Sur la géométrie du triangle*, in this Journal, vol. 11, 1917, p. 243.

For a system of coplaner n lines, n circles and n points may be associated in a variety of ways. The oldest one among the associations is that due to Clifford[1]. His theorem, which is a generalization of Theorem 5 can be enunciated as follows after Prof. Coolidge:

Theorem 6. *Given n lines in a plane, no two parallel and no three concurrent. If n be odd there is associated therewith a circle, and if n be even a point. The circle will contain the n points associated with the n sets of lines obtained by neglecting each of the given lines in turn; the point will lie on each of the n-circles obtained by neglecting each of the lines in turn.*

For our endeavour to prove this theorem by using complex numbers, Theorem 3 and its different forms given after its enunciation also play a great rôle. To accomplish the endeavour, we are wanted to prove Theorem 5 and Theorem 6 for five lines, since we can then arrive at the final result by repeatedly using Theorem 3 and its different forms.

We proceed to prove Theorem 5. If the given four lines be

$$(1) \qquad ABE, \qquad CDE, \qquad ADF, \qquad BCF,$$

then the four circles passing through the four sets of three points

$$(2) \qquad (A, B, F), \qquad (C, D, F), \qquad (A, D, E), \qquad (B, C, E)$$

respectively are concurrent.

Let the polar coordinates of the six points

$$A, \qquad B, \qquad C, \qquad D, \qquad E, \qquad F$$

be

$$(a, \alpha), \qquad (b, \beta), \qquad (c, \gamma), \qquad (d, \delta), \qquad (e, \varepsilon), \qquad (f, \varphi)$$

respectively, and let the inclinations of the four straight lines (1) with respect to the initial line be

$$\xi_1, \qquad \xi_2, \qquad \xi_3, \qquad \xi_4$$

respectively. Next let the polar coordinates of the centres of the four circles (2) be

$$(\rho_1, \theta_1), \qquad (\rho_2, \theta_2), \qquad (\rho_3, \theta_3), \qquad (\rho_4, \theta_4),$$

---

(1) Clifford, *A synthetic proof of Miquel's theorem*, Messenger of math., vol. 5. 1870. For the literature, see the foot-note on p. 90 of Prof. Coolidge's book and also consult Prof. F. Morley's paper, *On the metric geometry of the plane n–line*, Trans. of Amer. Math. Soc., vol. 1, 1900, p. 97.

and let their radii be

$$r_1, \qquad r_2, \qquad r_3, \qquad r_4$$

respectively.

For the points on the circumferences of the four circles (2), we have

$$(3) \qquad [\rho_1, \theta_1]+[r_1, \theta],$$
$$(4) \qquad [\rho_2, \theta_2]+[r_2, \theta],$$
$$(5) \qquad [\rho_3, \theta_3]+[r_3, \theta],$$
$$(6) \qquad [\rho_4, \theta_4]+[r_4, \theta],$$

θ being a variable angle, which is measured from the positive direction of the initial line. Since the circle (3) passes through the points A, B, F, at which we assume θ to take the values $\theta_a, \theta_b, \theta_f$ successively, we have

$$[\rho_1, \theta_1]=[a, \alpha]-[r_1, \theta_a]=[b, \beta]-[r_1, \theta_b]=[f, \varphi]-[r_1, \theta_f].$$

But evidently

$$\theta_a+\theta_b+\pi=2\xi_1,$$
$$\theta_b+\theta_f+\pi=2\xi_4,$$
$$\theta_f+\theta_a+\pi=2\xi_3,$$

whence

$$\theta_a=\xi_1+\xi_3-\xi_4-\pi/2,$$
$$\theta_b=\xi_4+\xi_1-\xi_3-\pi/2,$$
$$\theta_f=\xi_3+\xi_4-\xi_1-\pi/2.$$

Hence

$$\begin{aligned}[\rho_1, \theta_1]&=[a, \alpha]-[r_1, \eta_2-2\xi_4]\\ &=[b, \beta]-[r_1, \eta_2-2\xi_3]\\ &=[f, \varphi]-[r_1, \eta_2-2\xi_1],\end{aligned}$$

where

$$\eta_2=\xi_1+\xi_3+\xi_4-\pi/2.$$

For the sake of simplicity, transfer the origin of coordinates to the centre of circle (3). Then

$$\rho_1=0, \qquad a=b=f=r_1.$$

Hence we have

$$0=[1, \alpha]-[1, \eta_2-2\xi_4]$$

$$=[1, \beta]-[1, \eta_2-2\xi_3]$$
$$=[1, \gamma]-[1, \eta_2-2\xi_1].$$

Therefore for the points of intersection, not A, B and F, of the circles (5), (6) and (4) with the circle (3), we have

$$[r_1,\ 2\theta_3-\eta_2+2\xi_4], \qquad [r_1,\ 2\theta_4-\eta_2+2\xi_2], \qquad [r_1,\ 2\theta_2-\eta_2+2\xi_1]$$

respectively. Therefore for our purpose we must prove that

$$\theta_3+\xi_4=\theta_4+\xi_3=\theta_2+\xi_1.$$

Now the following relations can be proved by quite the same way as (7):

$$[\rho_3, \theta_3]=[a, \alpha]-[r_3, \eta_4-2\xi_2],$$
$$[\rho_4, \theta_4]=[b, \beta]-[r_4, \eta_3-2\xi_2],$$
$$[\rho_2, \theta_2]=[f, \varphi]-[r_2, \eta_1-2\xi_2],$$

whence

$$[1, 2(\theta_3+\xi_4)]=\mathfrak{A}/\overline{\mathfrak{A}},$$
$$[1, 2(\theta_4+\xi_3)]=\mathfrak{B}/\overline{\mathfrak{B}},$$
$$[1, 2(\theta_2+\xi_1)]=\mathfrak{C}/\overline{\mathfrak{C}},$$

where

$$\mathfrak{A}=[r_3, \xi]-[r_1, \xi_1+\xi_3], \qquad \overline{\mathfrak{A}}=[r_3, -\xi]-[r_1, -(\xi_1+\xi_3)],$$
$$\mathfrak{B}=[r_4, \xi]-[r_1, \xi_4+\xi_1], \qquad \overline{\mathfrak{B}}=[r_4, -\xi]-[r_1, -(\xi_4+\xi_1)],$$
$$\mathfrak{C}=[r_2, \xi]-[r_1, \xi_3+\xi_4], \qquad \overline{\mathfrak{C}}=[r_2, -\xi]-[r_1, -(\xi_3+\xi_4)],$$

and

$$\xi=\xi_1-\xi_2+\xi_3+\xi_4;$$

so that we are now in the position to prove that

$$\mathfrak{A}/\overline{\mathfrak{A}}=\mathfrak{B}/\overline{\mathfrak{B}}=\mathfrak{C}/\overline{\mathfrak{C}}.$$

If we simplify the first part of these equations, we arrive at

$$r_1 \sin(\xi_3-\xi_4)+r_4 \sin(\xi_4-\xi_2)+r_3 \sin(\xi_2-\xi_3)=0,$$

which expresses the collinearity of the three points A, B, E. Similar for the others.

4. Now let us proceed to prove Theorem 6 for five lines, that is, the five Miquel points for the five sets of the four lines, neglecting one of them in turn, lie on one and the same circle.

Let the points of intersection of the four lines ABE, CDE, ADF,

BCF with the fifth line be K_1, K_2, K_3, K_4 respectively, and let the polar coordinates of the five Miquel points M_1, M_2, M_3, M_4 and M_5 got by neglecting the first, second, third, fourth and fifth lines successively be (m_1, μ_1), $(m_2, \mu_2), \cdots\cdots, (m_5, \mu_5)$ respectively. Then the six sets of four points

$$(M_1, M_2,\ K_3, K_4),$$
$$(M_1, M_3,\ K_2, K_4),$$
$$(M_1, M_4,\ K_2, K_3),$$
$$(M_2, M_3,\ K_1, K_4),$$
$$(M_2, M_4,\ K_1, K_3),$$
$$(M_3, M_4,\ K_1, K_2)$$

are concyclic (the points F, C, D, B, A, E lying also on these circles respectively)(1).

With respect to the sixth, third, first and fourth circles

$$(K_1, K_2,\ M_3, M_4),$$
$$(K_2, K_3,\ M_4, M_1),$$
$$(K_3, K_4,\ M_1, M_2),$$
$$(K_4, K_1,\ M_2, M_3),$$

we know that the four points K_1, K_2, K_3, K_4 lie on one and the same straight line. Therefore the other four points M_1, M_2, M_3, M_4 must lie on one and the same circle, according to Theorem 3.

Similarly the four points M_1, M_2, M_3, M_5 lie on one and the same circle.

Thus the theorem is completely proved.

5. The equation formed by equating the determinant (1) in Article 1 to zero may be regarded as the equation to the rectangular hyperbola referred to rectangular coordinate system, passing through the three points (α_1', β_1'), (α_2, β_2) and (α_3, β_3), and having the coordinate axes as its asymptotes. Hence the words "circles" in the enunciation of Theorem 3 can be replaced by the words "rectangular hyperbolas," understanding that all the rectangular hyperbolas under consideration have parallel asymptotes.

Thus by projection we can arrive at the following theorem.

Theorem 7. *If the conics passing through the three points*

(1) Compare with one of the transformations of Theorem 3 in Art. 3.

$$(P_1', P_2, P_3), \qquad (P_2', P_3, P_1), \qquad (P_3', P_1, P_2)$$

respectively pass through a common point, the three conics passing through the three points

$$(P_1, P_2', P_3'), \qquad (P_2, P_3', P_1'), \qquad (P_3, P_1', P_2')$$

respectively also pass through a common point, all the conics having parallel asymptotes, or passing through two fixed points.

December 1917.

A Construction-Problem in Elementary Projective Geometry,

by

Tsuruichi Hayashi, Sendai.

Suggested by the problem proposed by Prof. T. Kubota on page 162 of this Journal vol. 13, mentioned in Japanese(1), I am in this note to solve the following construction-problem: *Given five point-pairs*

$$O_1, O_2;\ A_1, A_2;\ B_1, B_2;\ C_1, C_2;\ D_1, D_2$$

on a plane, to find the sixth point-pair X_1, X_2 *on the same plane, such that four conics pass through the four six-point sets*

$$(1)\qquad \begin{cases} O_1, O_2, A_1, A_2, X_1, X_2; \\ O_1, O_2, B_1, B_2, X_1, X_2; \\ O_1, O_2, C_1, C_2, X_1, X_2; \\ O_1, O_2, D_1, D_2, X_1, X_2. \end{cases}$$

During the solution, some note-worthy theorems, especially on two tetrads of points on a plane, are obtained, and in fine, a very simple method of construction of the conics defined by five points is given.

1. Suppose the required point-pair X_1, X_2 as obtained. Project the four four-point sets

$$A_1, A_2, X_1, X_2; \qquad B_1, B_2, X_1, X_2;$$
$$C_1, C_2, X_1, X_2; \qquad D_1, D_2, X_1, X_2,$$

from O_1 and O_2 on any straight line taken at random on the given plane, and denote the projections from O_1 by the same symbols as the projected points adding one dash and those from O_2 by those symbols adding two dashes. Then if we denote the anharmonic ratio of the range of four points P, Q, R, S by $(PQRS)$, the problem under consideration is reduced to the following: *To determine the two point-pairs* $X_1', X_2'; X_1'', X_2''$, *so that*

(1) For Prof. Kubota's problem itself, see the latter part of this note.

$$(A_1' A_2' X_1' X_2') = (A_1'' A_2'' X_1'' X_2''),$$
$$(B_1' B_2' X_1' X_2') = (B_1'' B_2'' X_1'' X_2''),$$
$$(C_1' C_2' X_1' X_2') = (C_1'' C_2'' X_1'' X_2''),$$
$$(D_1' D_2' X_1' X_2') = (D_1'' D_2'' X_1'' X_2''),$$

all the points lying on one and the same straight line.

This having been solved, the original problem can be solved by projecting back the range of points.

2. Let the distances of these points measured from any fixed point O on the straight line be denoted by the corresponding small letters. The last four relations can be transformed into the following equations

$$\frac{(a_1'-x_1')(a_2'-x_2')}{(a_2'-x_1')(a_1'-x_2')} = \frac{(a_1''-x_1'')(a_2''-x_2'')}{(a_2''-x_1'')(a_1''-x_2'')}, \text{ etc.,}$$

which can be again transformed into the following determinantal equations[1]

$$(2) \qquad \begin{vmatrix} x_1' x_1'' & x_1' & x_1'' & 1 \\ x_2' x_2'' & x_2' & x_2'' & 1 \\ a_1' a_1'' & a_1' & a_1'' & 1 \\ a_2' a_2'' & a_2' & a_2'' & 1 \end{vmatrix} = 0, \text{ etc..}$$

Hence algebraically stated, we are *to solve the four algebraical equations* (2).

3. Take any rectangular coordinate system (x', x'') on another plane, mark the points whose coordinates are

$$(a_1', a_1''),\ (a_2', a_2'');\ (b_1', b_1''),\ (b_2', b_2'');$$
$$(c_1', c_1''),\ (c_2', c_2'');\ (d_1', d_1''),\ (d_2', d_2''),$$

respectively, and denote these points by

$$A_1, A_2; \qquad B_1, B_2; \qquad C_1, C_2; \qquad D_1, D_2$$

respectively, which are yet different from those ten points denoted by the same letters in (1). Then since

$$\begin{vmatrix} x' x'' & x' & x'' & 1 \\ p' p'' & p' & p'' & 1 \\ q' q'' & q' & q'' & 1 \\ r' r'' & r' & r'' & 1 \end{vmatrix} = 0$$

(1) Burnside and Panton, Theory of Equations, vol. 2, 1904. p. 55.

represents a rectangular hyperbola passing through the three points $(p,' p'')$, (q', q''), (r', r'') whose asymptotes are parallel to the coordinate axes, we are to solve the problem: *Given four point-pairs*

$$A_1, A_2; \qquad B_1, B_2; \qquad C_1, C_2; \qquad D_1, D_2$$

on a plane, to find the fifth point-pair X_1, X_2 *on the same plane, such that four rectangular hyperbolas with parallel asymptotes pass through the four four-point sets*

$$A_1, A_2, X_1, X_2;$$
$$B_1, B_2, X_1, X_2;$$
$$C_1, C_2, X_1, X_2;$$
$$D_1, D_2, X_1, X_2.$$

If this particular case of the main problem be solved, the latter is also solved, because the process passed over is quite reversible.

4. Now let us treat in this and following articles conics *with parallel asymptotes*, if we say simply conics, taking off the limitation. Such two conics have only two *finite* intersections, real or imaginary, the join of which we may call the *radical axis* of the two conics. If such three conics be given, we have three radical axes by selecting two in turn among the three conics, passing through one and the same point, which we may call the *radical centre* of the three conics.

Consider the system of conics $\mathfrak{A}$ (with parallel asymptotes of course as we have remarked above) passing through two fixed points A_1, A_2, and cut them by any other conic S (whose asymptotes are also parallel to those of the system $\mathfrak{A}$). Then the radical axis of any one of the system and the latter conic S meet the join $A_1 A_2$ of the two fixed points at one and the same point, M_1 say.

Take another system of conics $\mathfrak{B}$ (whose asymptotes are all parallel to those of the conics previously considered), passing through other two fixed points B_1, B_2. Then the radical axis of any one of this second system and the conic S meet the join $B_1 B_2$ at one and the same point, N_1 say.

If the straight line $M_1 N_1$ be drawn, its two intersections with the conic S lie on one conic of the first system $\mathfrak{A}$ and also on the other conic of the second system $\mathfrak{B}$. The three conics have thus one and the same radical axis. Let us call this straight line the *radical axis of* S *with respect to* $\mathfrak{A}$ *and* $\mathfrak{B}$.

Let S be one member belonging to the third system of conics $\mathfrak{C}$, passing through the two fixed points C_1, C_2, and let it be changed from one of the system to the other. Then we get a set of radical axes of $\mathfrak{C}$ with respect to $\mathfrak{A}$ and $\mathfrak{B}$, which envelops a certain curve. As to this curve, we can prove the following theorem: *The set of radical axes of $\mathfrak{C}$ with respect to $\mathfrak{A}$ and $\mathfrak{B}$ envelops a conic, touching the base-lines $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ of the three systems.*

The set of radical axes of $\mathfrak{C}$ with respect to $\mathfrak{A}$ and $\mathfrak{B}$ is nothing but the set of radical axes of $\mathfrak{A}$ with respect to $\mathfrak{B}$ and $\mathfrak{C}$ or that of $\mathfrak{B}$ with respect to $\mathfrak{C}$ and $\mathfrak{A}$. So the set may be called the *set of radical axes of the three systems* $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$. Using this denomination, this theorem can be enunciated in the form: *The set of radical axes of the three systems of conics envelops a conic touching their base-lines.* The enveloping conic may be called the *radical conic of the three systems.*

5. Let us proceed to prove this theorem.

Referred to a rectangular coordinate system (x', x''), conics with parallel asymptotes may be represented by the equation

$$ax'^2 + 2hx'x'' + bx''^2 + 2gx' + 2fx'' + c = 0,$$

a, h, b being constant and g, f, c variable. If all the conics of this system pass through the fixed points $P \equiv (p', p'')$ and $Q \equiv (q', q'')$, two of the variables are determined. Let the two be g and f. Then if $f(x', x'')$ stands for $ax'^2 + 2hx'x'' + bx''^2$, the equation becomes

$$\begin{vmatrix} f(x', x'') + c & x' & x'' \\ f(p', p'') + c & p' & p'' \\ f(q', q'') + c & q' & q'' \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} f(x', x'') & x' & x'' \\ f(p', p'') & p' & p'' \\ f(q', q'') & q' & q'' \end{vmatrix} = -c \cdot \begin{vmatrix} x' & x'' & 1 \\ p' & p'' & 1 \\ q' & q'' & 1 \end{vmatrix},$$

where c remains variable, so that for one value of c corresponds one conic of the system.

If c becomes infinite, this reduces to

$$\begin{vmatrix} x' & x'' & 1 \\ p' & p'' & 1 \\ q' & q'' & 1 \end{vmatrix} = 0,$$

which represents the base-line PQ of the system. Hence *the set of the base-line and the line at infinity is to be regarded as a degenerate conic belonging to the systems.* Hence it follows that *the envelope of the radical axes of the three systems of conics $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ touch their base-lines $A_1 A_2$, $B_1 B_2$, $C_1 C_2$.*

6. Let the equation to a conic S belonging to $\mathfrak{A}$ be

$$(3) \qquad f(x', x'') + 2\beta x' + 2\alpha x'' + \gamma = 0,$$

in which

$$f(x', x'') = ax'^2 + 2hx' x'' + bx''^2,$$

and α and β are determined by the coordinates of the base-points A_1 and A_2 of the system. Then since the equation to any one of the conics belonging to $\mathfrak{B}$ is

$$(4) \qquad \begin{vmatrix} f(x', x'') & x' & x'' \\ f(b_1', b_1'') & b_1' & b_1'' \\ f(b_2', b_2'') & b_2' & b_2'' \end{vmatrix} = -c \begin{vmatrix} x' & x'' & 1 \\ b_1' & b_2'' & 1 \\ b_1' & b_1'' & 1 \end{vmatrix},$$

the coordinates of the radical centre of $\mathfrak{B}$ and S are given by

$$\begin{vmatrix} x' & x'' & 1 \\ b_1' & b_1'' & 1 \\ b_2' & b_2'' & 1 \end{vmatrix} = 0$$

and

$$\begin{vmatrix} f(x', x'') & x' & x'' \\ f(b_1', b_1'') & b_1' & b_1'' \\ f(b_2', b_2'') & b_2' & b_2'' \end{vmatrix} = \begin{vmatrix} b_1' & b_1'' \\ b_2' & b_2'' \end{vmatrix} \cdot \{ f(x', x'') + 2\beta x' + 2\alpha x'' + \gamma \},$$

and are found by solving these equations as simultaneous in the form

$$x' = \frac{k'\gamma + l'}{m'\gamma + n'}, \qquad x'' = \frac{k''\gamma + l''}{m''\gamma + n''},$$

where k, l, m, n are all independent of γ.

By quite the same way the coordinates of the radical centre of $\mathfrak{C}$ and S are found in similar forms in terms of γ.

Now we are searching the envelope of the joins of such point-pairs by changing the value of γ. Evidently the point-pairs form two homographic ranges on the base-line $B_1 B_2$ of $\mathfrak{B}$ and on the base-line $C_1 C_2$ of $\mathfrak{C}$. Hence the join of the corresponding points must envelop

the conic, which we have proposed above to call the radical conic of the three systems $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$.

7. Returning to the solution of the problem regarding the rectangular hyperbolas in Art. 3, we trace first the radical conic of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ and next that of $\mathfrak{A}, \mathfrak{B}, \mathfrak{D}$. Then these two conics have the two straight lines $A_1 A_2$ and $B_1 B_2$ as their common tangents. Draw the other two common tangents of the two radical conics. Then the required point-pair $(X_1 X_2)$ lies on each of the common tangents. Hence we have two solutions for the problem.

Now the method of construction is evident. Draw some of rectangular hyperbolas belonging to the four systems $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$. Determine the two radical conics, each by three tangents, having $A_1 A_2$ and $B_1 B_2$ as two common tangents, and draw the other two common tangents by the well known method(1). Then the problem is reduced to the construction of two conics, one belonging to $\mathfrak{A}$, and the other to $\mathfrak{B}$, intersecting on one of the two tangents, which is easily solved by the method of constructing the self-corresponding points of two superposed projective ranges(2).

8. Thus two straight lines are found from given four straight lines, all lying in one plane; or two point-pairs are found from given four point-pairs, all lying in one plane. This fact is quite similar to the existence of two straight lines intersecting four straight lines given in space. Indeed using this theorem in Space-geometry Prof. T. Kubota has proved by stereographical projection the existence of four coaxal circles passing through given four point-pairs respectively, and he has given a note containing the proposal to construct the four circles by operations in the plane only, when the arrangement of manuscripts and other materials for volume 13 of this Journal is going on(3). Just at that time, I have been studying some properties of circle-systems and have begun to solve a more general problem than Prof. Kubota's one, as mentioned in Art. 1 in this paper. When I have informed him some of the results obtained by me, he has been induced and has succeeded to solve his own problem and my extended problem. His solution is quite the same as that which I have mentioned in Art. 7 in this paper,

---

(1) Cremona, Projective Geometry. 1893, p. 190.

(2) Since the asymptotes in Arts. 4–6 may be imaginary, we can solve Prof. Kubota's problem for circles by a similar method.

(3) Prof. Kubota's solution has been published in Japanese thereafter in this Journal, vol. 13, p. 248, while his problem is found in vol. 13, p. 162.

but his demonstration has still remained stereographical, though very elegant. So continuing my own process of attack, I have arrived at the same solution as his, by using considerations in Plane Geometry only, though somewhere analytical([1]). However it must be informed to the reader that my work has been completed by his favour of having shown his solution, though it is quite similar to the method by which I have ever proved Mr. Kakeya's theorem in this Journal vol. 2, p. 213, as is shown in the next article([2]).

9. By using the well known theorem that there are two straight lines intersecting four straight lines given in space, of which Prof. T. Kubota has also taken advantage to prove his theorem for coaxal circles above mentioned, Mr. S. Kakeya has proved the theorem that there are in general two point-pairs X_1, X_2; $X_1' X_2'$ for given four point-pairs

$$A_1, A_2; \quad B_1, B_2; \quad C_1, C_2; \quad D_1, D_2,$$

such that

$$X_1 A_1 || X_2 A_2, \quad X_1 B_1 || X_2 B_2, \quad X_1 C_1 || X_2 C_2, \quad X_1 D_1 || X_2 D_2,$$

and

$$X_1' A_1 || X_2' A_2, \quad X_1' B_1 || X_2' B_2, \quad X_1' C_1 || X_2' C_2, \quad X_1' D_1 || X_2' D_2.$$

Denote the four systems of parallel straight lines passing through the four point-pairs by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{D}$ respectively. Take a fixed pair of parallel lines S belonging to $\mathfrak{A}$. Then all the joins of the points of intersection of any pair of parallel lines belonging to $\mathfrak{B}$ with S pass through a fixed point on $B_1 B_2$. Similar for the system $\mathfrak{C}$. If the join of these two fixed points on $B_1 B_2$ and $C_1 C_2$ be drawn, the points of intersection of the join with S are the common points of one of $\mathfrak{B}$, one of $\mathfrak{C}$, and S. When we change S in $\mathfrak{A}$, we get the envelope of such joins, that is a conic touching $A_1 A_2, B_1 B_2, C_1 C_2$. So to show the remaining part of the proof should be now more than enough.

In quite the same manner as Mr. Kakeya has deduced a double-six theorem from his theorem, we can deduce some double-six theorems from our theorems obtained in this paper.

([1]) The analytical parts may be reproduced by synthetical treatment without difficulty.

([2]) I have proposed a construction-problem arising from Mr. Kakeya's theorem in Japanese in this Journal vol. 3, p. 127.

By using Mr. Kakeya's theorem, we can prove the following theorem: *There are in general two point-pairs* X_1, X_2; X_1', X_2' *for given four point-pairs*

$$A_1, A_2; \quad B_1, B_2; \quad C_1, C_2; \quad D_1, D_2,$$

such that

$$X_1 A_1, X_2 A_2; \quad X_1 B_1, X_2 B_2; \quad X_1 C_1, X_2 C_2; \quad X_1 D_1, X_2 D_2$$

are antiparallel with respect to a given straight line and

$$X_1' A_1, X_2' A_2; \quad X_1' B_1, X_2' B_2; \quad X_1' C_1, X_2' C_2; \quad X_1' D_1, X_2' D_2$$

are also antiparallel with respect to the same straight line, and we can construct the two point-pairs, by taking the symmetrical points of the second points of the given four point-pairs with respect to the given straight line.

10. We have seen in article 3 that the conics, the construction of which was the main object of this note, were replaced by the rectangular hyperbolas with parallel asymptotes. The same idea can be used to get *a very simple and practical method of construction of the conic defined by given five points.*

Let P, Q, A, B, C be the given five points, and D be the fifth point on the conic passing through the five points. Draw any two mutually perpendicular straight lines AX and AY through the point A, and project the points B, C, and D from the points P and Q on these straight lines respectively. Denote the projections by

$$B', B''; \quad C', C''; \quad D', D''.$$

Then the three points which have these three point-pairs as the feet of their rectangular coordinates with respect to the two straight lines must lie on a rectangular hyperbola passing through the point A and having the two straight lines as its asymptotes. However it may be, we must have

$$(A, B', C', D') \barwedge (A, B'', C'', D''),$$

and therefore

$$(B', C', D') \,\overline{\barwedge}\, (B'', C'', D'').$$

Hence $D'D''$ must pass through the point of intersection of $B'B''$ and

$C' C''$. Hence the construction: draw $B' B''$ and $C' C''$, and draw any straight line $D' D''$ passing through their point of intersection; then the point of intersection of PD' and QD'' is the required point D.

A similar method may be used very advantageously for the construction of the fourth common point of the two conics, of which one passes through the five points P, Q, R, A, B, and the other through the five points P, Q, R, C, D.

December 1917.

Über die Schwerpunkte der konvexen geschlossenen Kurven und Flächen,

von

TADAHIKO KUBOTA in Sendai.

Im Folgenden behandle ich das nachstehende Problem und dessen Analogon im Raume: Was ist die notwendige und hinreichende Bedingung dafür, dass der Schwerpunkt einer konvexen geschlossenen Kurve mit den der konvexen Parallelkurven zusammenfällt? Zuerst stellen wir uns die Frage: Was ist der geometrische Ort der Schwerpunkte der konvexen Parallelkurven einer gegebenen konvexen geschlossenen Kurve? (1)

Es seien

$$x=x(s),\ \ y=y(s)\ (^2)$$

die Definitionsgleichungen der gegebenen konvexen geschlossenen Kurve, wobei der Parameter s die Kurvenlänge im positiven Umlaufe bedeutet. Bezeichnet man die ganze Kurvenlänge mit L, so gelten die Relationen:

$$\begin{aligned} x\ (s+L)&=x\ (s), & y\ (s+L)&=y\ (s),\\ x'\,(s+L)&=x'\,(s), & y'\,(s+L)&=y'\,(s),\\ x''(s+L)&=x''(s), & y''(s+L)&=y''(s). \end{aligned}$$

Ferner bezeichne man die Richtungskosinus der Tangente mit

$$a_1=\frac{dx}{ds},\ \ a_2=\frac{dy}{ds},$$

und die der nach aussen gerichteten Normale mit

$$c_1=\frac{dy}{ds},\ \ c_2=-\frac{dx}{ds}.$$

---

(1) Konstruiert man auf allen Seiten eines konvexen Polygons die ähnlich und ähnlich-gelegenen Dreiecke nach aussen, so ist der Schwerpunkte der neuen Eckepunkte mit dem der Eckepunkte des ursprünglichen Polygons koinzident. (Eine Verallgemeinerung des Satzes 352 in F.G.M., Exercises de Géométrie, 1912). Als ich diesen Satz ferner auf die konvexe geschlossene Fläche zu verallgemeinern versuchte, wurde ich zur obigen Erforschung veranlasst.

(2) Wir setzen voraus, dass alle vorkommenden Funktionen zweimal stetig differenzierbar sind.

Dann kann man die Parameterdarstellung der Parallelkurve wie folgt setzen:

$$\xi = x(s) + hc_1(s),$$
$$\eta = y(s) + hc_2(s),$$

wobei h eine gewisse Konstante und ξ, η laufende Koordinaten bedeuten.

Differenziert man die beiden Gleichungen nach s, so erhält man nach der Frenet-Serretschen Formel

$$\frac{d\xi}{ds} = \frac{dx}{ds} + h\frac{dc_1}{ds} = a_1\left(1 + \frac{h}{r}\right),$$
$$\frac{d\eta}{ds} = \frac{dy}{ds} + h\frac{dc_2}{ds} = a_2\left(1 + \frac{h}{r}\right),$$

wobei $r(s)$ den Krümmungsradius der Kurve im Kurvenpunkte s bedeutet. Bezeichnet man nun die Kurvenlänge der konvexen Parallelkurve mit $\bar{s}$, sodass s und $\bar{s}$ zueinander entsprechen, so ergibt sich die Relation:

$$\left(\frac{d\bar{s}}{ds}\right)^2 = \left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2 = \left(1 + \frac{h}{r}\right)^2, \ d.\,h. \ \frac{d\bar{s}}{ds} = 1 + \frac{h}{r}.$$

Folglich bekommt man für die Koordinaten (l, m) der Schwerpunkte der konvexen Parallelkurve den folgenden Ausdruck:

$$l = \int\left(x + hc_1\right)\left(1 + \frac{h}{r}\right) ds : \int\left(1 + \frac{h}{r}\right) ds$$
$$= \left(\int x ds + h\int c_1 ds + h\int\frac{x}{r} ds + h^2\int\frac{c_1}{r} ds\right) \div \left(\int ds + h\int\frac{ds}{r}\right),$$
$$m = \left(\int y ds + h\int c_2 ds + h\int\frac{y}{r} ds + h^2\int\frac{c_2}{r} ds\right) \div \left(\int ds + h\int\frac{ds}{r}\right),$$

wobei die Integration auf die ganze Kurve zu erstrecken ist. Wenn man nun einen Vektor parallel zur Normale des Kurvenpunktes s aus dem Koordinatenanfangspunkte zieht, so trifft dieser Vektor den Einheitskreis mit dem Zentrum $(0, 0)$ im Punkte mit den Koordinaten $c_1(s)$, $c_2(s)$. Also wird der Schwerpunkt des Einheitskreises durch

$$\frac{1}{2\pi}\int\frac{c_1(s)}{r(s)} ds, \quad \frac{1}{2\pi}\int\frac{c_2(s)}{r(s)} ds$$

gegeben. Da der Schwerpunkt dieses Einheitskreises natürlicherweise der Anfangspunkt ist, so gelten die Relationen:

$$\int\frac{c_1}{r} ds = 0, \quad \int\frac{c_2}{r} ds = 0,$$

und ausserdem haben wir

$$\int c_1 ds = \int \frac{dy}{ds} ds = 0, \quad \int c_2 ds = 0, \quad \int \frac{ds}{r} = 2\pi,$$

da die gegebene Kurve nach der Voraussetzung konvex und geschlossen ist.

Folglich

$$l = \left\{ \int x ds + h \int \frac{x}{r} ds \right\} \div (L + 2\pi h),$$

$$m = \left\{ \int y ds + h \int \frac{y}{r} ds \right\} \div (L + 2\pi h).$$

Somit bekommt man folgendes Resultat.

Die Schwerpunkte der konvexen Parallelkurven einer konvexen geschlossenen Kurve liegen auf einer Geraden.

Die Schwerpunkte der konvexen geschlossenen Kurven sind dann und nur dann mit dem Schwerpunkte der Originalkurve koinzident, wenn die Relationen

$$\frac{1}{L}\int x ds = \frac{1}{2\pi}\int \frac{x}{r} ds, \qquad \frac{1}{L}\int y ds = \frac{1}{2\pi}\int \frac{y}{r} ds$$

gelten.

Die Grössen $\frac{1}{2\pi}\int \frac{x}{r} ds$, $\frac{1}{2\pi}\int \frac{y}{r} ds$ sind aber die Koordinaten des sogenannten Krümmungsschwerpunktes (1) der gegebenen konvexen geschlossenen Kurve. Folglich bekommt man den folgenden Satz:

Die Schwerpunkte der konvexen Parallelkurven einer gegebenen konvexen geschlossenen Kurve sind dann und nur dann mit dem Schwerpunkte der Originalkurve koinzident, wenn der Schwerpunkt der gegebenen Kurve mit dem Krümmungsschwerpunkte der Kurve zusammenfällt. Wenn irgend eine konvexe Parallelkurve einer konvexen geschlossenen Kurve denselben Schwerpunkt wie die Originalkurve besitzt, so haben alle Parallelkurven denselben Schwerpunkt.

Es entsteht also eine Frage: welcher Bedingung muss die zweimal stetig differenzierbare Stützgeradenfunktion $p(\theta)$ der konvexen geschlossenen Kurve unterworfen sein, damit der Schwerpunkt mit dem Krümmungsschwerpunkte zusammenfällt, wobei θ den durch die Normale sowie die x-Achse enthaltenen Winkel bedeutet. $p(\theta)$ sei in die folgende Fouriersche Reihe entwickelbar:

$$p(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta).$$

(1) J. Steiner, Von dem Krümmungsschwerpunkte ebener Kurven. Crelle J. 21 1838).

Dann haben wir aus

$$x = p(\theta)\cos\theta - p'(\theta)\sin\theta,$$
$$y = p(\theta)\sin\theta + p'(\theta)\cos\theta$$

die nachstehenden Relationen:

$$x = a_1 + \frac{a_0}{2}\cos\theta + \frac{1}{2}\sum_{n=2}^{\infty}\Big[a_n\Big\{(n+1)\cos(n-1)\theta - (n-1)\cos(n+1)\theta\Big\}$$
$$+ b_n\Big\{(n+1)\sin(n-1)\theta - (n-1)\sin(n+1)\theta\Big\}\Big],$$

$$y = b_1 + \frac{a_0}{2}\sin\theta + \frac{1}{2}\sum_{n=2}^{\infty}\Big[a_n\Big\{-(n-1)\sin(n+1)\theta - (n+1)\sin(n-1)\theta\Big\}$$
$$+ b_n\Big\{(n-1)\cos(n+1)\theta + (n+1)\cos(n-1)\theta\Big\}\Big].$$

Daraus erhält man unmittelbar

$$\frac{1}{2\pi}\int_0^{2\pi} x\,d\theta = a_1, \qquad \frac{1}{2\pi}\int_0^{2\pi} y\,d\theta = b_1,$$

folglich *sind* a_1, b_1 *die Koordinaten des Steinerschen Krümmungsschwerpunktes der Kurve.*

Wenn man von vornherein den Steinerschen Krümmungschwerpunkt als Koordinatenanfangspunkt annimmt, so erhält man

$$a_1 = 0, \qquad b_1 = 0.$$

Dann

$$x = \frac{a_0}{2}\cos\theta + \frac{1}{2}\sum_{n=2}^{\infty}\Big[a_n\Big\{(n+1)\cos(n-1)\theta - (n-1)\cos(n+1)\theta\Big\}$$
$$+ b_n\Big\{(n+1)\sin(n-1)\theta - (n-1)\sin(n+1)\theta\Big\}\Big],$$

$$y = \frac{a_0}{2}\sin\theta + \frac{1}{2}\sum_{n=2}^{\infty}\Big[a_n\Big\{-(n-1)\sin(n+1)\theta - (n+1)\sin(n-1)\theta\Big\}$$
$$+ b_n\Big\{(n-1)\cos(n+1)\theta + (n+1)\cos(n-1)\theta\Big\}\Big].$$

Nun

$$\frac{ds}{d\theta} = \rho = p(\theta) + p''(\theta)$$
$$= \frac{a_0}{2} - \sum_{n=2}^{\infty}\Big[(n^2-1)a_n\cos n\theta + (n^2-1)b_n\sin n\theta\Big].$$

Die Bedingung dafür, dass der Schwerpunkt der Kurve mit dem Krümmungsschwerpunkt zusammenfällt, ist

$$\int x ds = 0, \quad \int y ds = 0,$$

d.h.
$$\int_0^{2\pi} x(p+p'')d\theta = 0, \quad \int_0^{\pi} y(p+p'')d\theta = 0.$$

Daraus erhält man als gewünschte Bedingung

$$\sum_{n=2}^{\infty}(n+2)(n-1)(a_n a_{n+1} + b_n b_{n+1}) = 0,$$

$$\sum_{n=2}^{\infty}(n+2)(n-1)(a_n b_{n+1} - a_{n+1} b_n) = 0.$$

Schliesslich gelangen wir zu dem Satze:

Die notwendige und hinreichende Bedingung dafür, dass die Schwerpunkte der konvexen Parallelkurven einer gegebenen konvexen Kurve mit dem Schwerpunkte der ursprünglichen Kurve zusammenfallen, besteht darin, dass, wenn die Stützgeradenfunktion $p(\theta)$ der C'' Klasse mit dem Krümmungsschwerpunkt als Anfangspunkt in die Fouriersche Reihe

$$p(\theta) = \frac{a_0}{2} + \sum_{n=2}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

entwickelbar ist, die Relationen

$$\sum_{n=2}^{\infty}(n+2)(n-1)(a_n a_{n+1} + b_n b_{n+1}) = 0,$$

$$\sum_{n=2}^{\infty}(n+2)(n-1)(a_n b_{n+1} - a_{n+1} b_n) = 0$$

gelten.

Für die Kurven konstanter Breite mit zweimal stetig differenzierbarer Stützgeradenfunktion ist die obige Bedingung erfüllt, da

$$a_{2n} = 0, \ b_{2n} = 0. \qquad (n = 1, 2, 3 \cdots\cdots).$$

Somit bekommen wir als unmittelbare Folge unseres Hauptsatzes:

Alle konvexen Parallelkurven einer Kurve konstanter Breite mit der Stützgeradenfunktion der C'' Klasse haben denselben Schwerpunkt.

Bei der Kurve konstanter Breite ist der Schwerpunkt mit dem Krüm-

mungsschwerpunkt der Kurve koinzident. Der letzte Satz wurde schon von Herrn E. Meissner (1) bewiesen.

Nun betrachten wir die analoge Aufgabe im Raume. Es sei eine konvexe geschlossene Fläche vorgelegt und seien

$$x=x(u, v), \ y=y(u, v), \ z=z(u, v)$$

die Definitionsgleichungen der Fläche und $X(u, v)$, $Y(u, v)$, $Z(u, v)$ seien die Richtungskosinus der nach aussen gerichteten Normale im Flächenpunkte (u, v). So sind

$$\xi=x(u, v)+hX(u, v), \ \eta=y(u, v)+hY(u, v), \ \zeta=z(u, v)+hZ(u, v)$$

die Definitionsgleichungen der konvexen Parallelfläche der gegebenen Fläche, wobei h eine gewisse Konstante bedeutet. Bezeichnet man den elementaren Flächeninhalt der gegebenen Fläche mit dS und den entsprechenden Flächeninhalt der Parallelfläche mit $d\bar{S}$, so werden die Koordinaten (l, m, n) des Schwerpunktes der Parallelfläche durch die nachstehenden Gleichungen gegeben:

$$l=\iint(x+hX)d\bar{S} : \iint d\bar{S},$$

$$m=\iint(y+hY)d\bar{S} : \iint d\bar{S},$$

$$n=\iint(z+hZ)d\bar{S} : \iint d\bar{S},$$

wobei die Integration auf die ganze Fläche zu erstrecken ist. Für die Koordinaten des Schwerpunktes der Originalfläche bekommt man

$$\iint xdS : \iint dS, \qquad \iint ydS : \iint dS, \qquad \iint zdS : \iint dS.$$

Bezeichnet man den Flächeninhalt der sphärischen Abbildung der Normalen, welcher dem Flächeninhalt dS oder $d\bar{S}$ entspricht, mit $d\sigma$, so ergeben sich die Relationen nach Gauss (2)

$$dS=R_1 R_2 \, d\sigma, \qquad d\bar{S}=(R_1+h)(R_2+h)d\sigma,$$

wobei R_1, R_2 die beiden Hauptkrümmungsradien im Flächenpunkte (u, v) und R_1+h, R_2+h die der Parallelfläche bedeuten. Daraus erhält man wie beim vorigen Probleme

(1) Meissner, Über die Anwendung von Fourier-Reihen auf einige Aufgaben der Geometrie und Kinematik, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich 54, 1909. Diese Schrift ist mir leider nicht zugänglich. Ich habe durch Schillingsche Arbeit kennen gelernt, dass dieser Satz schon von Meissner bewiesen wurde. F. Schilling, Die Theorie u. Konstruktion der Kurve konstanter Breite, Zeitschrift für Math. u. Physik, 1914.

(2) Gauss, Disquisitiones Generales circa Superficies Curvas, Werke 4.

$$l=\iint(x+hX)(R_1+h)(R_2+h)d\sigma \ : \ \iint(R_1+h)(R_2+h)d\sigma,$$

$$m=\iint(y+hY)(R_1+h)(R_2+h)d\sigma \ : \ \iint(R_1+h)(R_2+h)d\sigma,$$

$$n=\iint(z+hZ)(R_1+h)(R_2+h)d\sigma \ : \ \iint(R_1+h)(R_2+h)d\sigma.$$

Da aber X, Y, Z als die Koordinaten der sphärischen Abbildung der Flächennormale im Punkte (u, v) auf der Einheitskugel mit dem Zentrum $(0, 0, 0)$ betrachtet werden können, so sind

$$\frac{1}{4\pi}\iint Xd\sigma,\ \frac{1}{4\pi}\iint Yd\sigma,\ \frac{1}{4\pi}\iint Zd\sigma$$

die Koordinaten des Schwerpunktes der Einheitskugel. Also bekommt man die Relationen

$$\iint Xd\sigma=0,\ \iint Yd\sigma=0,\ \iint Zd\sigma=0,$$

und

$$\iint d\sigma=4\pi.$$

Folglich gelangt man zu den folgenden Relationen:

$$l=\left[\iint xR_1\,R_2\,d\sigma+h\iint\left\{XR_1\,R_2+x(R_1+R_2)\right\}d\sigma+h^2\iint\left\{x+X(R_1+R_2)\right\}d\sigma\right]$$
$$\div\left\{\iint R_1\,R_2\,d\sigma+h\iint(R_1+R_2)d\sigma+4\pi h^2\right\},$$

$$m=\left[\iint yR_1\,R_2\,d\sigma+h\iint\left\{YR_1\,R_2+y(R_1+R_2)\right\}d\sigma+h^2\iint\left\{y+Y(R_1+R_2)\right\}d\sigma\right]$$
$$\div\left\{\iint R_1\,R_2\,d\sigma+h\iint(R_1+R_2)d\sigma+4\pi h^2\right\},$$

$$n=\left[\iint zR_1\,R_2\,d\sigma+h\iint\left\{ZR_1\,R_2+Z(R_1+R_2)\right\}d\sigma+h^2\iint\left\{z+Z(R_1+R_2)\right\}d\sigma\right]$$
$$\div\left\{\iint R_1\,R_2 d\sigma+h\iint(R_1+R_2)d\sigma+4\pi h^2\right\},$$

wobei

$$\iint R_1\,R_2\,d\sigma=\text{dem Flächeninhalte der gegebenen Fläche.}$$

Wenn h sich stetig ändert, so beschreibt der Punkt (l, m, n) einen Kegelschnitt. Daraus folgt der Satz:

Die Schwerpunkte der konvexen Parallelflächen einer gegebenen konvexen geschlossenen Fläche liegen auf einem Kegelschnitte.

Wenn irgend zwei verschiedene konvexe Parallelflächen einer konvexen geschlossenen Fläche denselben Schwerpunkt wie die Originalfläche besitzen, so haben wir notwendigerweise

$$\frac{\iint xR_1 R_2\, d\sigma}{\iint R_1 R_2\, d\sigma} = \frac{\iint [XR_1 R_2 + x(R_1+R_2)]d\sigma}{\iint (R_1+R_2)d\sigma} = \frac{\iint [x+X(R_1+R_2)]d\sigma}{4\pi},$$

$$\frac{\iint yR_1 R_2\, d\sigma}{\iint R_1 R_2\, d\sigma} = \frac{\iint [YR_1 R_2 + y(R_1+R_2)]d\sigma}{\iint (R_1+R_2)d\sigma} = \frac{\iint [y+Y(R_1+R_2)]d\sigma}{4\pi},$$

$$\frac{\iint zR_1 R_2\, d\sigma}{\iint R_1 R_2\, d\sigma} = \frac{\iint [ZR_1 R_2 + z(R_1+R_2)]d\sigma}{\iint (R_1+R_2)d\sigma} = \frac{\iint [z+Z(R_1+R_2)]d\sigma}{4\pi}.$$

Folglich haben alle konvexen Parallelflächen denselben Schwerpunkt. Somit bekommen wir den Satz:

Wenn irgend zwei verschiedene konvexe Parallelflächen einer konvexen geschlossenen Fläche denselben Schwerpunkt wie die Originalfläche haben, so besitzen alle konvexen Parallelflächen denselben Schwerpunkt.

Sendai, den 14ten Februar, 1918.

Theory of the Point-Line Connex (1, 1) in Space, I,

by

KINNOSUKE OGURA, Ôsaka.

Clebsch(1) established the theory of *the point-line connex in a plane.* For example, if x_1, x_2, x_3 be the homogeneous point coordinates and u_1, u_2, u_3 the homogeneous line coordinates, then

$$(a_1 u_1 + b_1 u_2 + c_1 u_3) x_1 + (a_2 u_1 + b_2 u_2 + c_2 u_3) x_2 + (a_3 u_1 + b_3 u_2 + c_3 u_3) x_3 = 0,$$

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

defines a principal coincidence (1, 1) in the plane. The principal coincidence (1, 1) leads us to a quadratic duality of the points and lines in the plane; and also to a system of W–curves which are nothing but the curves of the coincidence.

The theory of *the point-plane connex in space* has been developed in similar ways(2); on the contrary, that of *the point-line connex in space,* as far as I am aware, is remained almost untouched(3). So in this paper I propose to make a study of the point-line connex (1, 1) in space.

Chapter I deals with the single principal point-line coincidence (1, 1) in space. This coincidence has a close connection with the principal point-line coincidence (1, 1) in a plane, and with the reciprocal of Steiner's roman surface.

The point corresponding to a line is, in general, determined uniquely. Some properties of the aggregate of the lines which make the corresponding points indeterminate, together with a contact transforma-

---

(1) Clebsch, Math. Ann., 6 (1873), p. 203; Clebsch-Lindemann, Vorlesungen über Geometrie, I (1876), p. 924; Fouret, Bull. de la Soc. Math. France, 2 (1874), p. 72.

(2) Fouret, Comptes Rend. Paris, 80 (1875), p. 167; R. Krause, Math. Ann., 14 (1879), p. 294; Lazzeri, Mem. Accad. Lincei, (4) 4 (1887), p. 259; Autonne, Sur les formes mixtes (1905).

(3) For the suggestions, see Clebsch, Über eine Fundamentalaufgabe der Invariantentheorie, Math. Ann., 5 (1872). p. 427; Klein, Einleitung in die höhere Geometrie, I, 2. Aufl. (1907), p. 253; p. 253; and especially Ogura, A geometrical study of the mechanics of a particle, this Journal, 13 (1918), p. 172.

tion, will be discussed in Chapter II. In this place we have several points of contact with the investigations of Kummer (for the line congruence (2, 3)), R. Krause (for the point-plane connex (2, 1)), Prof. Reye (for the linear system of ∞^3 quadrics) and especially Prof. Voss (for the point-plane system)(1).

Chapter III deals with the simultaneous principal point-line coincidences (1, 1) in space. We can establish *a (1, 1) correspondence between the lines of a certain cubic complex Γ and the points of space.* At the end some examples of special varieties of this correspondence are mentioned; especially the example I (§27) gives a natural extension of the principal coincidence (1, 1) in a plane.

In the paper following to this, I expect to treat the curves of simultaneous principal coincidences and some problems of differential-geometric side.

CHAPTER I.

The principal point-line coincidence (1, 1) in space.

1. Let x_1, x_2, x_3, x_4 be the homogeneous point coordinates and $p_{12}, p_{23}, p_{31}, p_{41}, p_{42}, p_{43}$ the homogeneous line (radial) coordinates. Then the bilinear equation

$$(1) \qquad P_1 x_1 + P_2 x_2 + P_3 x_3 + P_4 x_4 = 0,$$

where

$$P_1 \equiv a_{12}\,p_{12} + a_{23}\,p_{23} + a_{31}\,p_{31} + a_{41}\,p_{41} + a_{42}\,p_{42} + a_{43}\,p_{43},$$
$$P_2 \equiv b_{12}\,p_{12} + b_{23}\,p_{23} + b_{31}\,p_{31} + b_{41}\,p_{41} + b_{42}\,p_{42} + b_{43}\,p_{43},$$
$$P_3 \equiv c_{12}\,p_{12} + c_{23}\,p_{23} + c_{31}\,p_{31} + c_{41}\,p_{41} + c_{42}\,p_{42} + c_{43}\,p_{43},$$
$$P_4 \equiv d_{12}\,p_{12} + d_{23}\,p_{23} + d_{31}\,p_{31} + d_{41}\,p_{41} + d_{42}\,p_{42} + d_{43}\,p_{43},$$

all a, b, c, d being constants, defines a *point-line connex of the first order and the first class.*

The totality of all the elements of the connex (1) which satisfy the *condition of incidence*

$$(2) \qquad -p_{43}x_2 + p_{42}x_3 + p_{23}x_4 = 0,$$
$$(3) \qquad p_{43}x_1 \qquad - p_{41}x_3 + p_{31}x_4 = 0,$$

(1) See also R. Sturm, Über höhere räumliche Nullsysteme, Math. Ann., 28 (1887), p. 277.

$$(4) \qquad -p_{42}x_1+p_{41}x_2 \qquad +p_{12}x_4=0,$$
$$(5) \qquad -p_{23}x_1-p_{31}x_2-p_{12}x_3 \qquad =0$$

will be called *the principal point-line coincidence of the first order and the first class.*

We will denote the space by Σ_p when the line (p) is given, and by Σ_x when the point (x) is given.

The space Σ_p.

2. When a line (p) is given, equation (1) denotes a plane which will be called E_p. The point corresponding to (p) in the principal coincidence is determined as the intersection of the given line (p) and the plane E_p.

Since there exists the identity

$$p_{12}p_{43}+p_{23}p_{41}+p_{31}p_{42}=0,$$

the determinant

$$\begin{vmatrix} 0 & -p_{43} & p_{42} & p_{23} \\ p_{43} & 0 & -p_{41} & p_{31} \\ -p_{42} & p_{41} & 0 & p_{12} \\ -p_{23} & -p_{31} & -p_{12} & 0 \end{vmatrix}$$

is of rank 2; hence we obtain from (1), (2), (3) *the point corresponding to* (p):

$$(6) \qquad \begin{cases} \rho x_1= \qquad -P_2p_{12}+P_3p_{31}+P_4p_{41}\equiv\phi_1, \\ \rho x_2= \quad P_1p_{12} \qquad -P_3p_{24}+P_4p_{42}\equiv\phi_2, \\ \rho x_3=-P_1p_{31}+P_2p_{23} \qquad +P_4p_{43}\equiv\phi_3, \\ \rho x_4=-P_1p_{41}-P_2p_{42}-P_3p_{43} \qquad \equiv\phi_4, \end{cases}$$

ρ being the proportional factor.

The totality of all points corresponding to all lines in the space Σ_p forms the (ordinary) space in general. (Compare with § 7).

3. If a given line (p) belong to the figure (*the singular figure in the space Σ_p*) defined by

$$(7) \qquad \phi_1=\phi_2=\phi_3=\phi_4=0,$$

the corresponding point becomes indeterminate. These equations (7) express the condition that the line (p) should be contained in the plane E_p.

But it is seen that the following identities exist:

$$-p_{23}\,\phi_1-p_{31}\,\phi_2\equiv p_{12}\,\phi_3,$$
$$p_{42}\,\phi_1-p_{41}\,\phi_2\equiv p_{12}\,\phi_4.$$

We have, then, two cases:

(i) $p_{12}\neq 0$, $\phi_3=0$, $\phi_4=0$ for the whole intersection $(\phi_{1,2})$ or the partial intersection $[\phi_{1,2}]$ of the two complexes $\phi_1=0$ and $\phi_2=0$;

(ii) $p_{12}=0$ for $(\phi_{1,2})$.

In the former the corresponding point is indeterminate when the line (p) belongs to $(\phi_{1,2})$ or $[\phi_{1,2}]$ respectively. In the latter, in order that the corresponding point should be indeterminate, we must have

$$p_{12}=0,\quad \phi_3=0,\quad \phi_4=0.$$

But by the identities

$$p_{41}\,\phi_3-p_{31}\,\phi_4\equiv p_{43}\,\phi_1,$$
$$p_{42}\,\phi_3+p_{23}\,\phi_4\equiv p_{43}\,\phi_2,$$

two cases arise:

(i) $p_{43}\neq 0$, $\phi_1=0$, $\phi_2=0$ for $(\phi_{3,4})$ or $[\phi_{3,4}]$,

(ii) $p_{43}=0$ for $(\phi_{3,4})$.

Thus we obtain the theorem: I. *The singular figure consists of* $(\phi_{1,2})$ *or* $[\phi_{1,2}]$, *if* $(\phi_{1,2})$ *or* $[\phi_{1,2}]$ *be not contained in the complex* $p_{12}=0$. (*This may be replaced by* $(\phi_{3,4})$ *or* $[\phi_{3,4}]$, *if* $(\phi_{3,4})$ *or* $[\phi_{3,4}]$ *be not contained in the complex* $p_{34}=0$); II. *The singular figure is the linear congruence* $p_{12}=p_{34}=0$, *if* $(\phi_{1,2})$ *be contained in* $p_{12}=0$ *and* $(\phi_{3,4})$ *in* $p_{34}=0$.

Now we can prove the theorem: *In general*(¹), *the singular figure is a congruence of the second order and third class.*

Firstly the number of lines of the singular figure which lie in any given plane is 3. To prove this, we may take $x_4=0$ as a given plane without any loss of generality(²). The lines of the singular figure in the plane $x_4=0$ are given by

$$\phi_1=\phi_2=\phi_3=\phi_4=0,$$

(¹) Throughout this paper the word *in general* is used in the sense that no special connections exist among the constants a, b, c, d (and a', b', c', d' in § 18).

(²) In general any property of the principal coincidence is independent of the choice of the tetrahedron of reference.

where we put

$$p_{41}=p_{42}=p_{43}=0,$$
$$p_{23}=u_1, \quad p_{31}=u_2, \quad p_{12}=u_3,$$

that is,

$$\frac{a_{23}\,u_1+a_{31}\,u_2+a_{12}\,u_3}{u_1}=\frac{b_{23}\,u_1+b_{31}\,u_2+b_{12}\,u_3}{u_2}$$
$$=\frac{c_{23}\,u_1+c_{31}\,u_2+c_{12}\,u_3}{u_3}.$$

Since u_1, u_2, u_3 are the line coordinates in the plane $x_4=0$, the above system of equations gives three lines.

Secondly the number of lines the singular figure which pass through any given point is 2. To prove this, we may take $(0, 0, 0, 1)$ as a given point without any loss of generality. The lines of the singular figure passing through the point $(0, 0, 0, 1)$ are given by

$$\phi_1=\phi_2=\phi_3=\phi_4=0,$$

where we put

$$p_{12}=p_{23}=p_{31}=0,$$
$$p_{41}=-x_1, \quad p_{42}=-x_2, \quad p_{43}=-x_3;$$

that is,

$$\begin{cases} d_{41}\,x_1+d_{42}\,x_2+d_{43}\,x_3=0, \\ (a_{41}\,x_1+a_{42}\,x_2+a_{43}\,x_3)\,x_1+(b_{41}\,x_1+b_{42}\,x_2+b_{43}\,x_3)\,x_2 \\ \qquad\qquad +(c_{41}\,x_1+c_{42}\,x_2+c_{43}\,x_3)\,x_3=0, \end{cases}$$

which gives two lines. (Compare with § 7.)

The space Σ_x.

4. *When a point (x) is given, the ∞^1 corresponding lines of the principal coincidence form a flat pencil having that point as the vertex.* If we put

$$(8) \qquad A_{ik}=a_{ik}\,x_1+b_{ik}\,x_2+c_{ik}\,x_3+d_{ik}\,x_4, \quad (i, k=1, 2, 3, 4; \quad i\neq k),$$

and

$$(9) \qquad \begin{cases} U_1= \qquad\qquad -A_{12}\,x_2+A_{31}\,x_3+A_{41}\,x_4, \\ U_2= \quad A_{12}\,x_1 \qquad\qquad -A_{23}\,x_3+A_{42}\,x_4, \\ U_3=-A_{31}\,x_1+A_{23}\,x_2 \qquad\qquad +A_{43}\,x_4, \\ U_4=-A_{41}\,x_4-A_{42}\,x_2-A_{43}\,x_3 \qquad\qquad , \end{cases}$$

the base E_x of the flat pencil is given by

$$(10) \qquad y_1 U_1 + y_2 U_2 + y_3 U_3 + y_4 U_4 = 0,$$

y_1, y_2, y_3, y_4 being the current point coordinates. This can be shown by substituting

$$p_{ik} = x_i y_k - y_i x_k$$

in equation (1).

5. If the line (p) corresponding to a point (x) lie in a plane E, the point (x) must be contained in that plane. Let (x) be any given point in a plane E. Then there is, at least, one line (p) corresponding to that point (x) and lying in the plane E; (p) is the intersection of the two planes E and E_x. Hence we have ∞^2 lines (p) corresponding to all points (x) of E and lying in E.

Now we can prove that *the set of* (x, p) *in the plane* E *is nothing but a principal point-line coincidence* $(1, 1)$ *in this plane.* For, in the general case we may take

$$y_4 = 0$$

as the plane E without any loss of generality. Then $x_4 = 0$, and hence

$$p_{41} = p_{42} = p_{43} = 0,$$

$$p_{23} = u_1, \quad p_{31} = u_2, \quad p_{12} = u_3,$$

u_1, u_2, u_3 being the line coordinates in the plane. Consequently (1) and (2)–(5) are reduced into

$$\begin{cases} (a_{23} u_1 + a_{31} u_2 + a_{12} u_3) x_1 + (b_{23} u_1 + b_{31} u_2 + b_{12} u_3) x_2 \\ \qquad\qquad + (c_{23} u_1 + c_{31} u_2 + c_{12} u_3) x_3 = 0, \\ u_1 x_1 + u_2 x_2 + u_3 x_3 = 0, \end{cases}$$

from which the theorem follows.

It follows from this result that all the lines, corresponding to points in Σ_x and passing through any fixed point, form a bundle. Therefore we arrive at the theorem :

The totality of all lines corresponding to all points in the space Σ_x forms, in general, the (ordinary) line space.

For the condition that the totality should form a complex, see § 7 below.

6. Consider the envelope of the plane E_x :

$$(10) \qquad U_1 y_1 + U_2 y_2 + U_3 y_3 + U_4 y_4 = 0,$$

(y_1, y_2, y_3, y_4 being the current point coordinates), when the point (x) describes the plane E:

(11) $$\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+\lambda_4 x_4=0, \qquad (\lambda_i \text{ const.}).$$

If we put

(12) $$z_i=U_i(x_1, x_2, x_2, x_4), \qquad (i=1, 2, 3, 4),$$

the point (z) describes a *Steiner* (*roman*) *surface* S_E when (x) describes the plane E(1). Then, since (10) becomes

$$z_1 y_1+z_2 y_2+z_3 y_3+z_4 y_4=0,$$

the plane E_x envelopes the reciprocal $\bar{S}_E$ of the *Steiner* surface S_E with respect to

$$y_1^2+y_2^2+y_3^2+y_4^2=0.$$

If we introduce the plane coordinates v_1, v_2, v_3, v_4, the surface $\bar{S}_E$ can be written parametrically(2):

(13) $$\begin{cases} v_i=U_i(x_1, x_2, x_3, x_4) \qquad (i=1, 2, 3, 4), \\ \lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3+\lambda_4 x_4=0, \end{cases}$$

which may be considered as the plane equations of the plane E_x.

Since the intersection of the two planes E and E_x is the line (p) corresponding to (x) and lying in E, we have, by aid of § 5, the theorem:

Consider a principal coincidence (1, 1) *in a plane, for example, in the plane* $x_4=0$ *defined by*

$$\begin{cases} (a_{23} u_1+a_{31} u_2+a_{12} u_3) x_1+(b_{23} u_1+b_{31} u_2+b_{12} u_3) x_2 \\ \qquad\qquad +(c_{23} u_1+c_{31} u_2+c_{12} u_3) x_3=0, \\ u_1 x_1+u_2 x_2+u_3 x_3=0; \end{cases}$$

and take the plane

(13′) $$\begin{cases} \rho v_1=(-a_{12}x_2+a_{31}x_3)x_1+(-b_{12}x_2+b_{31}x_3)x_2+(-c_{12}x_2+c_{31}x_3)x_3, \\ \rho v_2=(-a_{23}x_3+a_{12}x_1)x_1+(-b_{33}x_3+b_{12}x_1)x_2+(-c_{23}x_3+c_{12}x_1)x_3, \\ \rho v_3=(-a_{31}x_1+a_{23}x_2)x_1+(-b_{31}x_1+b_{23}x_2)x_2+(-c_{31}x_1+c_{23}x_2)x_3, \\ \rho v_4=-(a_{41}x_1+a_{42}x_2+a_{43}x_3)x_1-(b_{41}x_1+b_{42}x_2+b_{43}x_3)x_2 \\ \qquad\qquad -(c_{41}x_1+c_{42}x_2+c_{43}x_3)x_2, \end{cases}$$

(1) Reye, Geometrie der Lage, III, 3. Aufl. (1892), p. 147.

(2) This surface is often called the Cayley surface. See Pascal, Repertorium der höheren Mathematik, II, 1. Aufl. (1902), p. 279.

v_1, v_2, v_3, v_4 *being the plane coordinates and* $a_{41}, b_{41}, c_{41}, \cdots$ *arbitrary constants. Then the intersection of this plane and* $x_4=0$ *is nothing but the line* (n) *corresponding to the giving point* (x). *When* (x) *describes the plane* $x_4=0$, *the plane* $(13')$ *envelopes the surface* $\bar{S}_{(x_4=0)}$.

A special case.

7. *In order that the totality of all the lines corresponding to all the points in* Σ_x *should form a complex*, all the lines (p), lying in any plane and corresponding to all the points in that plane, must envelope a single curve; this case occurs, by the first theorem in § 5, when and only when all the above lines (p) form a pencil[1]; so that *the complex must be linear.*

When the totality of all the lines corresponding to all the points in Σ_x is a linear complex

$$\phi_0 \equiv A\, p_{12} + B\, p_{23} + C p_{31} + D\, p_{41} + E\, p_{42} + F p_{43} = 0,$$

the point (x) and the plane E_x must form the null system

$$\begin{gathered}(-A\, x_2 + C x_3 + D\, x_4)\, y_1 + (A\, x_1 - B\, x_3 + E\, x_4)\, y_2 \\ +(-C x_1 + B\, x_2 + F x_4)\, y_3 - (D\, x_1 + E\, x_2 + F x_3)\, y_4 = 0;\end{gathered}$$

whence

$$\begin{aligned} U_1 &\equiv (\qquad -A\, x_2 + C x_3 + D\, x_4)\, U_0, \\ U_2 &\equiv (A\, x_1 \qquad - B\, x_3 + E\, x_4)\, U_0, \\ U_3 &\equiv (-C\, x_1 + B\, x_2 \qquad + F x_4)\, U_0, \\ U_4 &\equiv (-D\, x_1 - E\, x_2 - F x_3 \qquad)\, U_0, \end{aligned}$$

U_0 being a linear form of x_1, x_2, x_3, x_4; that is, each of the four quadrics $U_i=0$ breaks up into two planes, one being common to all quadrics.

For example, if we take

$$U_0 \equiv x_4,$$

then

$$\begin{array}{llllll} a_{12}=0, & a_{23}=k, & a_{31}=0, & a_{41}=0, & a_{42}=a, & a_{43}=-c, \\ b_{12}=0, & b_{23}=0, & b_{31}=k, & b_{41}=-a, & b_{32}=0, & b_{43}=b, \\ c_{12}=k, & c_{23}=0, & c_{31}=0, & c_{41}=c, & c_{42}=-b, & c_{43}=0; \end{array}$$

$$A=a+d_{12}, \qquad B=b+d_{23}, \qquad C=c+d_{31},$$

---

[1] Clebsch-Lindemann, loc. cit., pp. 999, 1001.

$$D=d_{41}, \qquad E=d_{42}, \qquad F=d_{43};$$

so that the connex (1) is given by

$$\begin{aligned}
P_1 &= \qquad k\,p_{23} \qquad\qquad\qquad\qquad +a\,p_{42} - c\,p_{43},\\
P_2 &= \qquad\qquad k\,p_{31} - a\,p_{41} \qquad\qquad + b\,p_{43},\\
P_3 &= k\,p_{12} \qquad\qquad + c\,p_{41} \quad - b\,p_{42} \qquad\qquad ,\\
P_4 &= d_{12}\,p_{12} + d_{23}\,p_{23} + d_{31}\,p_{31} + d_{41}\,p_{41} + d_{42}\,p_{42} + d_{43}\,p_{43}.
\end{aligned}$$

Since

$$\phi_1 \equiv p_{41}\,\phi_0, \qquad \phi_2 \equiv p_{42}\,\phi_0, \qquad \phi_3 \equiv p_{43}\,\phi_0, \qquad \phi_4 \equiv 0,$$

the point corresponding to the line (p) is given by

$$\rho x_1 = p_{41}, \qquad \rho x_2 = p_{42}, \qquad \rho x_3 = p_{43}, \qquad \rho x_4 = 0;$$

that is, the point of intersection of (p) and the plane $x_4=0$. When the line (p) is contained in the plane $x_4=0$ or belongs to the complex $\phi_0=0$, the corresponding point becomes indeterminate; so that *the singular figure consists of the linear complex* $\phi_0=0$ *and the improper congruence* $p_{41}=p_{42}=p_{43}=0$.

The totality of all the points corresponding to all the lines (excluding the singular figure) in the space Σ_p *forms the plane* $x_4=0$.

CHAPTER II.

Relation between the two spaces Σ_p and Σ_x.

A contact transformation.

8. It can be easily proved that

$$(1) \qquad y_1\,U_1(x) + y_2\,U_2(x) + y_3\,U_3(x) + y_4\,U_4(x) = 0$$

defines a *contact transformation between the two spaces* (x) *and* (y):

Space (x)	Space (y)
a point (x),	a plane I_x,
a quadric I_y,	a point (y),
a plane E,	$\overline{S}_E$,

where I_x (or I_y) denotes the equation (1) in which x_i (or y_i) $(i=1, 2, 3, 4)$ are fixed[1].

If we regard y_1, y_2, y_3, y_4 as parameters the equation (1) denotes the set of ∞^3 quadrics. The vertices of the cones included in this set are given by

$$(14) \qquad \begin{cases} \Sigma y_i \dfrac{\partial U_i(x)}{\partial x_1}=0, & \Sigma y_i \dfrac{\partial U_i(x)}{\partial x_2}=0, \\ \Sigma y_i \dfrac{\partial U_i(x)}{\partial x_3}=0, & \Sigma y_i \dfrac{\partial U_i(x)}{\partial x_4}=0; \end{cases}$$

and the locus of the vertices is the *Jacobian:*

$$(15) \qquad J(x) \equiv \begin{vmatrix} \dfrac{\partial U_1}{\partial x_1} & \dfrac{\partial U_2}{\partial x_1} & \dfrac{\partial U_3}{\partial x_1} & \dfrac{\partial U_4}{\partial x_1} \\ \dfrac{\partial U_1}{\partial x_2} & \dfrac{\partial U_2}{\partial x_2} & \dfrac{\partial U_3}{\partial x_2} & \dfrac{\partial U_4}{\partial x_2} \\ \dfrac{\partial U_1}{\partial x_3} & \dfrac{\partial U_2}{\partial x_3} & \dfrac{\partial U_3}{\partial x_3} & \dfrac{\partial U_4}{\partial x_3} \\ \dfrac{\partial U_1}{\partial x_4} & \dfrac{\partial U_2}{\partial x_4} & \dfrac{\partial U_3}{\partial x_4} & \dfrac{\partial U_4}{\partial x_4} \end{vmatrix} = 0.$$

Equations (14) may be written

$$(16) \qquad \begin{cases} \Sigma x_i f_{1i}(y)=0, & \Sigma x_i f_{2i}(y)=0, \\ \Sigma x_i f_{3i}(y)=0, & \Sigma x_i f_{4i}(y)=0, \end{cases}$$

where

$$(17) \qquad f_{ik}(y)=f_{ki}(y)=\frac{\partial U_i(y)}{\partial y_k}+\frac{\partial U_k(y)}{\partial y_i}.$$

Eliminating x_1, x_2, x_3, x_4 from (16) we have the *symmetroid* Δ:

$$(18) \qquad \Delta(y) \equiv \begin{vmatrix} f_{11}(y) & f_{12}(y) & f_{13}(y) & f_{14}(y) \\ f_{21}(y) & f_{22}(y) & f_{23}(y) & f_{24}(y) \\ f_{31}(y) & f_{32}(y) & f_{33}(y) & f_{34}(y) \\ f_{41}(y) & f_{42}(y) & f_{43}(y) & f_{44}(y) \end{vmatrix} = 0.$$

The point (y) on the symmetroid is *associated* to the point (x) on the Jacobian.

---

(1) I_x is, of course, identical with E_x.

Now *the four quadrics*

$$U_1(x)=0,\ U_2(x)=0,\ U_3(x)=0,\ U_4(x)=0$$

have 5 points in common (the *elementary points* of the principal coincidence). For, since we have the identity

$$x_1 U_1(x)+x_2 U_2(x)+x_3 U_3(x)+x_4 U_4(x)\equiv 0,$$

the common points of $U_1=0$, $U_2=0$, $U_3=0$ lie either in the plane $x_4=0$ $(U_4\neq 0)$, or in the quadric $U_4=0$ $(x_4\neq 0)$, or on the conic $x_4=0$, $U_4=0$. But if we put $x_4=0$, then $U_1=0$, $U_2=0$, $U_3=0$ take the forms

$$\frac{x_2}{[A_{31}]_{x_4=0}}=\frac{x_3}{[A_{12}]_{x_4=0}},\quad \frac{x_3}{[A_{12}]_{x_4=0}}=\frac{x_1}{[A_{23}]_{x_4=0}},\quad \frac{x_1}{[A_{23}]_{x_4=0}}=\frac{x_2}{[A_{31}]_{x_4=0}}$$

respectively; so that these three conics have 3 points in common, that is, the points of intersection of the first two conics excluding the point $x_3=0$, $[A_{12}]_{x_4=0}=0$. Also it is seen that these common points do not lie on the conic $x_4=0$, $U_4=0$. But since $U_1=0$, $U_2=0$, $U_3=0$ intersect at 8 points, the four quadrics $U_1=0$, $U_2=0$, $U_3=0$, $U_4=0$ have 5 points in common.

Therefore it follows that *the Jacobian J has the 5 elementary points as nodes, and also 20 lines; the symmetroid Δ has 15 nodes*(1).

9. Now we prove the theorem: *The elementary points are nodes of the symmetroid Δ.*

Firstly, the elementary points lie on the symmetroid Δ. For, since we have by direct calculations the following identities:

$$(19)\qquad \begin{cases} \Sigma y_i f_{1i}(y)=-U_1(y), & \Sigma y_i f_{2i}(y)=-U_2(y), \\ \Sigma y_i f_{3i}(y)=-U_3(y), & \Sigma y_i f_{4i}(y)=-U_4(y), \end{cases}$$

for the elementary points

$$U_1(y)=U_2(y)=U_3(y)=U_4(y)=0,$$

it must be

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{vmatrix} = \frac{1}{y}\begin{vmatrix} y_1f_{11}+y_2f_{12}+y_3f_{13}+y_4f_{14} & f_{12} & f_{13} & f_{14} \\ y_1f_{21}+y_2f_{22}+y_3f_{23}+y_4f_{24} & f_{22} & f_{23} & f_{24} \\ y_1f_{31}+y_2f_{32}+y_3f_{33}+y_4f_{34} & f_{32} & f_{33} & f_{34} \\ y_1f_{41}+y_2f_{42}+y_3f_{43}+y_4f_{44} & f_{42} & f_{43} & f_{44} \end{vmatrix}=0.$$

(1) Jessop, Quartic surfaces with singular points (1916), p. 173.

Next, the elementary points (y_1, y_2, y_3, y_4) are nodes of the symmetroid. For, from (17)

$$\begin{vmatrix} \dfrac{\partial f_{11}}{\partial y_1} & f_{12} & f_{13} & f_{14} \\ \dfrac{\partial f_{21}}{\partial y_1} & f_{22} & f_{23} & f_{24} \\ \dfrac{\partial f_{31}}{\partial y_1} & f_{32} & f_{33} & f_{34} \\ \dfrac{\partial f_{41}}{\partial y_1} & f_{42} & f_{43} & f_{44} \end{vmatrix} = \begin{vmatrix} 0 & f_{12} & f_{13} & f_{14} \\ -a_{12} & f_{22} & f_{23} & f_{24} \\ a_{31} & f_{32} & f_{33} & f_{34} \\ a_{41} & f_{42} & f_{43} & f_{44} \end{vmatrix} = \frac{1}{y_1} \begin{vmatrix} -\dfrac{f_{11}}{2} & 0 & 0 & 0 \\ -a_{12} & f_{22} & f_{23} & f_{24} \\ a_{31} & f_{32} & f_{33} & f_{34} \\ a_{41} & f_{42} & f_{43} & f_{44} \end{vmatrix}$$

$$= -\frac{f_{11}}{2y_1} \begin{vmatrix} f_{22} & f_{23} & f_{24} \\ f_{32} & f_{33} & f_{34} \\ f_{42} & f_{43} & f_{44} \end{vmatrix};$$

similarly

$$\begin{vmatrix} f_{11} & \dfrac{\partial f_{12}}{\partial y_1} & f_{13} & f_{14} \\ f_{21} & \dfrac{\partial f_{22}}{\partial y_1} & f_{33} & f_{34} \\ f_{31} & \dfrac{\partial f_{32}}{\partial y_1} & f_{33} & f_{34} \\ f_{41} & \dfrac{\partial f_{42}}{\partial y_1} & f_{43} & f_{44} \end{vmatrix} = \frac{f_{12}}{2y_1} \begin{vmatrix} f_{21} & f_{23} & f_{24} \\ f_{31} & f_{33} & f_{34} \\ f_{41} & f_{43} & f_{44} \end{vmatrix}, \;\cdots\cdots, \;\cdots\cdots.$$

Consequently we obtain

$$\frac{\partial \Delta}{\partial y_1} \equiv \frac{\partial}{\partial y_1} \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{vmatrix} = -\frac{1}{2y_1} \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ f_{44} & f_{42} & f_{43} & f_{44} \end{vmatrix} = 0.$$

Similarly

$$\frac{\partial \Delta}{\partial y_2} = 0, \qquad \frac{\partial \Delta}{\partial y_3} = 0, \qquad \frac{\partial \Delta}{\partial y_4} = 0,$$

from which follows the theorem.

It is seen from (16) that there exists one-to-one correspondence

between associate points on J and Δ. Here we prove that *the 5 elementary points are self-associate.* For, we have from (16)

$$x_1 : x_4 = -\begin{vmatrix} f_{14} & f_{12} & f_{13} \\ f_{24} & f_{22} & f_{23} \\ f_{34} & f_{32} & f_{33} \end{vmatrix} : \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}.$$

The elementary points give from (19) that

$$\begin{vmatrix} f_{14} & f_{12} & f_{13} \\ f_{24} & f_{22} & f_{23} \\ f_{34} & f_{32} & f_{33} \end{vmatrix} = \frac{1}{y_4}\begin{vmatrix} f_{12}\,y_2+f_{13}\,y_3+f_{14}\,y_4 & f_{12} & f_{13} \\ f_{22}\,y_2+f_{23}\,y_3+f_{24}\,y_4 & f_{22} & f_{23} \\ f_{32}\,y_2+f_{33}\,y_3+f_{34}\,y_4 & f_{32} & f_{33} \end{vmatrix} = \frac{1}{y_4}\begin{vmatrix} -y_1 f_{11} & f_{12} & f_{13} \\ -y_1 f_{21} & f_{22} & f_{23} \\ -y_1 f_{31} & f_{32} & f_{33} \end{vmatrix};$$

so that

$$x_1 : x_4 = y_1 : y_4.$$

Similarly

$$x_2 : x_4 = y_2 : y_4, \qquad x_3 : x_4 = y_3 : y_4.$$

10. If we substitute the plane coordinates v_i in place of the point coordinates y_i or x_i, then

$$\text{(I)} \qquad y_1\,U_1(x)+y_2\,U_2(x)+y_3\,U_3(x)+y_4\,U_4(x)=0$$

becomes

$$\text{(II)} \qquad v_1\,U_1(x)+v_2\,U_2(x)+v_3\,U_3(x)+v_4\,U_4(x)=0$$

or

$$\text{(III)} \qquad y_1\,U_1(v)+y_2\,U_2(v)+y_3\,U_3(v)+y_4\,U_4(v)=0,$$

which represents a (particular) point-plane connex (2, 1) or (1, 2) respectively. Hence *the projective theorems concerning the contact transformation (I) can be deduced from those concerning the point-plane connex (2, 1) (II) or (1, 2) (III) by polar reciprocation.* The following are some typical examples(1):

(1) The following table shows the comparison of some terminologies and notations of the text, Reye (loc. cit.) and Krause (loc. cit).

	Steiner surface S	Its reciprocal $\overline{S}$	Jacobian J	Its reciprocal $\overline{J}$	Symmetroid Δ	Its reciprocal $\overline{\Delta}$
Reye	F_1^4	F^3	Kernfläche K^4			Brennfläche ϕ^4
Krause	ϕ		Kernfläche K			Determinanten-Fläche Δ

Contact transformation (I)	Point-plane connex (2,1) (II)	Point-plane connex (1,2) (III)
When the point (x) describes a plane (u), the plane I_x envelopes $\overline{S_u}$.	When the point (x) describes a plane (u), the point II_x describes S_u.	When the plane (v) passes through a point (x), the plane III_v envelopes $\overline{S_x}$(1).
When the point (x) describes a line, the plane I_x envelopes a quadric cone.	When the point (x) describes a line, the point II_x describes a conic.	When the plane (v) passes through a line, the plane III_v envelopes a quadric cone.
When the quadric I_y envelopes a plane (u), the point (y) describes $\overline{S_u}$.	When the quadric II_v envelopes a plane (u), the plane (v) envelopes S_u;	When the quadric III_y passes through a point (x), the point (y) describes $\overline{S_x}$.
The point (y), for which I_y becomes a cone, describes Δ; and the vertex (x) of the cone describes J. (x) and (y) are associated.	The plane (v), for which II_v becomes a cone, envelopes $\overline{\Delta}$; and the vertex (x) of the cone describes J. (x) and (v) are associated.	The point (y), for which III_y becomes a conic, describes Δ; and the plane (v) of the conic envelopes $\overline{J}$. (v) and (y) are associated.
When the plane (u) passes through a point (x) on the quadric I_y, $\overline{S_u}$ passes through the associate point of (x).	When the plane (u) passes through a point (x) on the quadric II_v, S_u touches the associate plane of (x).	When the point (x) lies in a tangent plane (u) of the quadric III_y, $\overline{S_x}$ passes through the associate point of (u).

The asymptotic lines of a certain point-plane system.

11. We have seen in § 4 all lines corresponding to a given point (x) form the pencil having (x) as the vertex and the plane E_x:

$$(10)\qquad y_1 U_1(x)+y_2 U_2(x)+y_3 U_3(x)+y_4 U_4(x)=0$$

as the base. Since there exists the identity

$$(20)\qquad x_1 U_1(x)+x_2 U_2(x)+x_3 U_3(x)+x_4 U_4(x)=0,$$

the point (x) *and the plane* E_x *form the point-plane system* (*of the second order*) of Prof. Voss(2).

Take any point $(\lambda x+\mu\xi)$ on the line joining any two points (x) and (ξ). Then the plane $E_{\lambda x+\mu\xi}$ takes the form

$$\Sigma y_i U_i(\lambda x+\mu\xi)=\Sigma y_i[\lambda^2 U_i(x)+\lambda\mu U_i(x\,|\,\xi)+\mu^2 U_i(\xi)],$$

(1) S_x is identical with S_u, when (x) is the pole of (u).

(2) Voss, Theorie der rationalen algebraischen Punkt-Ebenen-Systeme, Math. Ann., 23 (1884), p. 360.

where

$$U_i(x|\xi)=\sum_k \xi_k \frac{\partial U_i(x)}{\partial x_k}=\sum_k x_k \frac{\partial U_i(\xi)}{\partial \xi_k}.$$

In order that this plane should pass through the point (x), it is necessary and sufficient that

$$(21) \qquad \lambda \sum_i x_i \left[\sum_k \xi_k \frac{\partial U_i(x)}{\partial x_k} \right] + \mu \sum x_i U_i(\xi)=0,$$

since

$$(20) \qquad \sum x_i U_i(x)=0.$$

Differentiating (20) with respect to x_k,

$$x_1 \frac{\partial U_1(x)}{\partial x_k}+x_2 \frac{\partial U_2(x)}{\partial x_k}+x_3 \frac{\partial U_3(x)}{\partial x_k}+x_4 \frac{\partial U_4(x)}{\partial x_k}=-U_k(x),$$

so that (21) becomes

$$\lambda \sum \xi_i U_i(x)-\mu \sum x_i U_i(\xi)=0.$$

Therefore *the necessary and sufficient condition that the plane corresponding to every point of the line joining* (x) *and* (ξ) *pass through the point* (x) *is that* (x) *and* (ξ) *should satisfy the relations*

$$(22) \qquad \sum \xi_i U_i(x)=0$$

and

$$(23) \qquad \sum x_i U_i(\xi)=0.$$

When the condition is fulfilled, the above plane contains the line joining (x) *and* (ξ).

If the point (x) be taken as fixed, (22) denotes the tangent plane at (x) to the quadric (23); hence the line joining (x) and (ξ) is a generating line of the quadric (23).

The generating lines passing through (x) are nothing but *the asymptotic lines at* (x) *of the point-plane system.*

12. Prof. Voss([1]) proved that the totality of the asymptotic lines of the point-plane system (of the second order) forms a congruence $\mathfrak{C}$ of the second order and the third class. On the other hand, we have seen that the singular figure in the space Σ_p

$$\phi_1=\phi_2=\phi_3=\phi_4=0$$

([1]) Voss, loc. cit, p. 381.

is a congruence (2, 2). Here we can prove the theorem:

The singular figure in the space Σ_p is nothing but the congruence $\mathfrak{C}$.

For, the flat pencil formed by all the lines (p) corresponding to each point (x) of an asymptotic line contains the asymptotic line (§ 11); so that an asymptotic line corresponds to every point of it. But since this property is characteristic for the asymptotic line, our theorem has been proved.

In order to prove this analytically, we will find the line equations of the congruence $\mathfrak{C}$ consisting of the lines joining (x) and (ξ) which satisfy (22) and (23). By the identities

$$\begin{aligned}3\,U_i(x) &= 2\,U_i(x) + U_i(x)\\ &= \left[x_1\frac{\partial U_i(x)}{\partial x_1} + x_2\frac{\partial U_i(x)}{\partial x_2} + x_3\frac{\partial U_i(x)}{\partial x_3} + x_4\frac{\partial U_i(x)}{\partial x_4}\right]\\ &\quad - \left[x_1\frac{\partial U_1(x)}{\partial x_i} + x_2\frac{\partial U_2(x)}{\partial x_i} + x_3\frac{\partial U_3(x)}{\partial x_i} + x_4\frac{\partial U_4(x)}{\partial x_i}\right],\end{aligned}$$

(22) becomes

$$\sum_{i,k} y_{ik}\left(\frac{\partial U_i(x)}{\partial x_k} - \frac{\partial U_k(x)}{\partial x_i}\right) = 0, \quad (i, k = 1, 2, 3, 4), \tag{24}$$

where

$$p_{ik} = x_i\,\xi_k - \xi_i\,x_k\,.$$

Similarly we have from (23)

$$\sum_{i,k} p_{ik}\left(\frac{\partial U_i(\xi)}{\partial \xi_k} - \frac{\partial U_k(\xi)}{\partial \xi_i}\right) = 0. \tag{25}$$

Multiply (24) and (25) by ξ_1 and $-x_1$ respectively and add. Then remembering the definitions of U_i (§ 4), we arrive at the equation

$$\phi_1 = 0.$$

In similar ways we can obtain

$$\phi_2 = 0, \qquad \phi_3 = 0, \qquad \phi_4 = 0.$$

13. If the quadric (23) for a fixed point (x) becomes a cone, the two generating lines passing through the point (x) coincide, that is, the two asymptotic lines at (x) coincide; hence (x) must lie on the focal surface (in the sense of Kummer) of the congruence $\mathfrak{C}$. Since the converse is also true, the focal surface of the congruence $\mathfrak{C}$ is nothing but the symmetroid for the linear system of ∞^3 quadrics

$$x_1 U_1(\xi)+x_2 U_2(\xi)+x_3 U_3(\xi)+x_4 U_4(\xi)=0,$$

x_1, x_2, x_3, x_4 being considered as the homogeneous parameters. Thus we obtain the theorem:

The focal surface of the singular figure in the space Σ_p is the symmetroid

$$\Delta(x)=0. \tag{18'}$$

It is apparent that some of the properties of the symmetroid, obtained in §§ 8–9, are natural consequences of the general theory of Kummer(1) for the congruence (2, 3).

Now let (x) and (ξ) be any two given points on an asymptotic line (p). The theory of congruence teaches us that the asymptotic line (p) is a double tangent of the symmetroid (18′). Here we will give a direct proof of this by determining the points of contact. The pencil of quadrics

$$\Sigma(\lambda x_i+\mu\xi_i)U_i(y)=0 \tag{26}$$

corresponding to the range of points $(\lambda x+\mu\xi)$ on the asymptotic line (p) has the asymptotic line (p) and a fixed space cubic in common; for the two quadrics

$$\Sigma x_i\, U_i(y)=0 \quad \text{and} \quad \Sigma\xi_i\, U_i(y)=0$$

have the line (p) in common and consequently a space cubic also in common. The values of $\lambda:\mu$ which make the quadric (26) a cone are given by

$$\begin{vmatrix} \lambda m_{11}+\mu n_{11} & \lambda m_{12}+\mu n_{12} & \lambda m_{13}+\mu n_{13} & \lambda m_{14}+\mu n_{14} \\ \lambda m_{21}+\mu n_{21} & \lambda m_{22}+\mu n_{22} & \lambda m_{23}+\mu n_{23} & \lambda m_{24}+\mu n_{24} \\ \lambda m_{31}+\mu n_{31} & \lambda m_{32}+\mu n_{32} & \lambda m_{33}+\mu n_{33} & \lambda m_{34}+\mu n_{34} \\ \lambda m_{41}+\mu n_{41} & \lambda m_{42}+\mu n_{42} & \lambda m_{43}+\mu n_{43} & \lambda m_{44}+\mu n_{44} \end{vmatrix}=0, \tag{27}$$

where we have put

$$\sum_i x_i\, U_i(y)=\sum_{i,k} m_{ik}\, y_i\, y_k, \quad (m_{ik}=m_{ki}),$$

$$\sum_i \xi_i\, U_i(y)=\sum_{i,k} n_{ik}\, y_i\, y_k, \quad (n_{ik}=n_{ki}).$$

It is well known that in this case the equation (27) has two pairs of double roots (2). If any of these values $\lambda:\mu$ be given to $(\lambda x+\mu\xi)$, the

(1) Kummer, Über die algebraischen Strahlensysteme, Abh. Akad. Berlin, (1866), p. 1. See Voss, loc. cit.; Jessop, Treatise on the line complex (1903), p. 276.

(2) See, for example, Clebsch-Lindemann, Vorlesungen über Geometrie, II, 1 (1891), p. 221.

corresponding points lie on the symmetroid. Consequently the asymptotic line has the double contact with the symmetroid, and the points of contact are determined by solving the equation (27), the square of a certain quadratic equation.

14. Let (x) be a point on the symmetroid and (ξ) be the vertex of the cone corresponding to (x) (§ 13), that is, the associate to (x), on the Jacobian

$$(15') \qquad J(\xi)=0.$$

Since the line joining (x) and (ξ) is the generating line passing through (x), it is an asymptotic line. Hence *the line passing through any point* (x) *on the symmetroid and its associate* (ξ) *on the Jacobian belongs to the singular figure in* Σ_p.

To prove this analytically, consider the condition that (x) and (ξ) should be associated, i.e.

$$(14') \qquad \Sigma x_i \frac{\partial U_i(\xi)}{\partial \xi_1}=0,\ \Sigma x_i \frac{\partial U_i(\xi)}{\partial \xi_2}=0,\ \Sigma x_i \frac{\partial U_i(\xi)}{\partial \xi_3}=0,\ \Sigma x_i \frac{\partial U_i(\xi)}{\partial \xi_4}=0;$$

or

$$(16') \qquad \Sigma \xi_i f_{1i}(x)=0,\ \Sigma \xi_i f_{2i}(x)=0,\ \Sigma \xi_i f_{3i}(x)=0,\ \Sigma \xi_i f_{4i}(x)=0.$$

Multiplying four equations (14′) by ξ_1, ξ_2, ξ_3, ξ_4 respectively and adding, we have

$$(23) \qquad \Sigma x_i\, U_i(\xi)=0.$$

Similarly multiplying four equations (16′) by x_1, x_2, x_3, x_4 respectively and adding,

$$(22) \qquad \Sigma \xi_i\, U_i(x)=0.$$

Consequently the line joining (x) and (ξ) is an asymptotic line.

The principal lines of a certain linear system of ∞^3 quadrics.

15. It is well known that if (ξ) be a point on the Jacobian

$$J(\xi)=0,$$

its polar planes with respect to the four quadrics

$$U_1(y)=0,\ U_2(y)=0,\ U_3(y)=0,\ U_4(y)=0$$

meet in a point (x) on the Jacobian, and (ξ) and (x) are called *conjugate points on the Jacobian.*

Now the condition that (x) and (ξ) should be conjugate is given by

(28) $$\Sigma x_i \frac{\partial U_1(\xi)}{\partial \xi_i}=0,\ \Sigma x_i \frac{\partial U_2(\xi)}{\partial \xi_i}=0,\ \Sigma x_i \frac{\partial U_3(\xi)}{\partial \xi_i}=0,\ \Sigma x_i \frac{\partial U_4(\xi)}{\partial \xi_i}=0,$$

which may be written

(29) $$\Sigma \xi_i \frac{\partial U_1(x)}{\partial x_i}=0,\ \Sigma \xi_i \frac{\partial U_2(x)}{\partial x_i}=0,\ \Sigma \xi_i \frac{\partial U_3(x)}{\partial x_i}=0,\ \Sigma \xi_i \frac{\partial U_4(x)}{\partial x_i}=0.$$

Multiplying four equations (28) by ξ_1, ξ_2, ξ_3, ξ_4 respectively and adding, we obtain

(23) $$\Sigma x_i\, U_i(\xi)=0.$$

Similarly multiplying four equations (29) by x_1, x_2, x_3, x_4 respectively and adding,

(22) $$\Sigma \xi_i\, U_i(x)=0.$$

Hence the line joining any pair of conjugate points on the Jacobian belongs to the singular figure in Σ_p.

Such a line was called by Prof. Reye [1] *the principal line (Hauptstrahlen) of the linear system of* ∞^3 *quadrics*

$$\lambda_1\, U_1+\lambda_2\, U_2+\lambda_3\, U_3+\lambda_4\, U_4=0,$$

λ_i being parameters. Thus *in our case the asymptotic lines of Voss are equivalent to the principal lines of Reye.*

16. Let the point (z) describe the range

$$\rho z_i=\lambda x_i+\mu\xi_i\,,$$

whose base is the asymptotic line (p) joining two conjugate points (x) and (ξ) on the Jacobian. Then the plane E_z forms the axial pencil having the asymptotic lines as its axis, and the plane coordinates w_i of E_z are given by

$$\sigma w_i=\lambda^2 U_i(x)+\mu^2 U_i(\xi).$$

Consequently a plane (w) corresponds to the two points (z'), (z'') such that

(1) Reye, loc. cit., p. 144. Prof. Reye proved that if (x) and (ξ) be any pair of conjugate points and if we put

(12′) $$\rho z_i = U_i\,(x), \qquad (i=1, 2, 3, 4),$$
$$\sigma\zeta_i = U_i\,(\xi),$$

then the congruence consisting of the lines which join the corresponding pairs of (z) and (ζ) has the reciprocal of the symmetroid (§ 10) as the focal surface.

$$\rho z'_i = \lambda x_i + \mu \xi_i ,$$
$$\rho z''_i = \lambda x_i - \mu \xi_i .$$

Therefore *when the plane E_z describes the axial pencil, the pair of points (z'), (z'') forms an involution whose double points are the conjugates (x), (ξ) on the Jacobian.*

If the asymptotic lines at (z') be (p), (p') and those at (z'') be (p), (p''), then these three asymptotic lines (p), (p), (p'') lie on the plane $E_{z'}$ (which is the same as $E_{z''}$). Conversely, there are three asymptotic lines, (p), (p'), (p'') say, in any given plane (§ 3); if any two, for example (p'), (p''), of these lines intersect the third line (p) at (z'), (z'') respectively, the given plane will correspond to both (z') and (z''). Hence *in order that (p') and (p'') may coincide, it is necessary and sufficient that (z') or (z'') should lie on the Jacobian.*

17. Let Δ be the point of the symmetroid associate to any given point J of the Jacobian, and let J_1 be the point of the Jacobian conjugate to J. Then $J\Delta$ and JJ_1 are different lines in general.

Now we consider the line $J\Delta$. This line touches the symmetroid at Δ and another point $\bar{\Delta}$, and passes through the point $\bar{J}$ associate to $\bar{\Delta}$. The two other intersections J', J'_1 of this line and the Jacobian are conjugate to each other.

Next we come to the other line JJ_1. Let this line intersect the Jacobian at other two points $J_,''$ $\bar{J}_,''$ and touch the symmetroid at two points $\Delta_,''$ $\bar{\Delta}.''$ Then $J_,''$ $\bar{J}''$ are associates of $\Delta_,''$ $\bar{\Delta}.''$

CHAPTER III.

Simultaneous principal coincidences (1, 1).

A (1, 1) correspondence between the lines of the cubic complex Γ and the points of space.

The space Σ_p.

18. Now we proceed to consider the set of the elements common to the *two* principal coincidences defined by

$$(30) \qquad \begin{cases} P_1\, x_1 + P_2\, x_2 + P_3\, x_3 + P_4\, x_4 = 0, \\ P'_1\, x_1 + P'_2\, x_2 + P'_3\, x_3 + P'_4\, x_4 = 0, \end{cases}$$

where

$$\left\{\begin{aligned} P'_1&=a'_{12}p_{12}+a'_{23}p_{23}+a'_{31}p_{31}+a'_{41}p_{41}+a'_{42}p_{42}+a'_{43}p_{43},\\ P'_2&=b'_{12}p_{12}+b'_{23}p_{23}+b'_{31}p_{31}+b'_{41}p_{41}+b'_{42}p_{42}+b'_{43}p_{43},\\ P'_3&=c'_{12}p_{12}+c'_{23}p_{23}+c'_{31}p_{31}+c'_{41}p_{41}+c'_{42}p_{42}+c'_{43}p_{43},\\ P'_4&=d'_{12}p_{12}+d'_{23}p_{23}+d'_{31}p_{31}+d'_{41}p_{41}+d'_{42}p_{42}+d'_{43}p_{43}\end{aligned}\right.$$

under the condition of incidence (2)–(5).

When a line (p) is given, there can not exist, in general, any point corresponding to the given line. In order that the corresponding point may exist, (30) and any two of (2)–(5), for example (2) and (3), should be consistent for the same value of x_i. Hence the required condition is

$$\begin{vmatrix} P_1 & P_2 & P_3 & P_4 \\ P'_1 & P'_2 & P'_3 & P'_4 \\ 0 & -p_{43} & p_{42} & p_{23} \\ p_{43} & 0 & -p_{41} & p_{31} \end{vmatrix}=0.$$

Using the identity

$$p_{12}p_{43}+p_{23}p_{41}+p_{31}p_{42}=0$$

and dividing by the factor p_{43}, the above equation becomes

$$(31)\qquad \Gamma\equiv p_{12}\begin{vmatrix} P_1 & P_2 \\ P_1' & P_2' \end{vmatrix}+p_{23}\begin{vmatrix} P_2 & P_3 \\ P_2' & P_3' \end{vmatrix}+p_{31}\begin{vmatrix} P_3 & P_1 \\ P_3' & P_1' \end{vmatrix}$$
$$+p_{41}\begin{vmatrix} P_4 & P_1 \\ P_4' & P_1' \end{vmatrix}+p_{42}\begin{vmatrix} P_4 & P_2 \\ P_4' & P_2' \end{vmatrix}+p_{43}\begin{vmatrix} P_4 & P_3 \\ P_4' & P_3' \end{vmatrix}=0,$$

which may be written

$$(32)\qquad \begin{aligned}\Gamma&\equiv P_1'\,\phi_1+P_2'\,\phi_2+P_3'\,\phi_3+P_4'\,\phi_4\\ &\equiv(P_1\,\phi_1'+P_2\,\phi_2'+P_3\,\phi_3'+P_4'\,\phi_4')=0\ (^1).\end{aligned}$$

Therefore we have the theorem: *In order that there may exist the point corresponding to a line* (p), *it is necessary and sufficient that the given line must belong to the cubic complex* Γ.

This may be seen otherwise as follows: If the point (x) corresponding to a given line (p) exist, the two planes E_p and E'_p:

$$E_p:\qquad P_1\,x_1+P_2\,x_2+P_3\,x_3+P_4\,x_4=0,$$
$$E_p':\qquad P'_1\,x_1+P_2'\,x_2+P_3'\,x_3+P_4'\,x_4=0$$

must have the point (x) in common with (p); in other words, the intersection (p') of the two planes E_p and E'_p must intersect (p).

(1) For the definitions of ϕ'_i, see (35) below.

But the axial coordinates (π_{ik}') and radial coordinates (p_{ik}') of the line (p') have the expressions:

$$(33)\quad \begin{cases} \rho\pi_{12}'=\sigma p_{43}'=\begin{vmatrix} P_1 & P_2 \\ P_1' & P_2' \end{vmatrix}, & \rho\pi_{23}'=\sigma p_{41}'=\begin{vmatrix} P_2 & P_3 \\ P_2' & P_3' \end{vmatrix}, \quad \rho\pi_{31}'=\sigma p_{42}'=\begin{vmatrix} P_3 & P_1 \\ P_3' & P_1' \end{vmatrix}, \\ \rho\pi_{41}'=\sigma p_{23}'=\begin{vmatrix} P_4 & P_1 \\ P_4' & P_1' \end{vmatrix}, & \rho\pi_{42}'=\sigma p_{31}'=\begin{vmatrix} P_4 & P_2 \\ P_4' & P_2' \end{vmatrix}, \quad \rho\pi_{43}'=\sigma p_{12}'=\begin{vmatrix} P_4 & P_3 \\ P_4' & P_3' \end{vmatrix}, \end{cases}$$

ρ, σ being proportional factors. Hence the condition for the intersection of (p) and (p')

$$p_{12}\,p_{43}'+p_{23}\,p_{41}'+p_{31}\,p_{42}'+p_{41}\,p_{23}'+p_{42}\,p_{31}'+p_{43}\,p_{12}'=0$$

becomes

$$\Gamma=0.$$

19. When a line (p) belongs to the cubic complex Γ, the point (x) corresponding to (p) is the point of intersection of (p) and (p'). Hence

$$\begin{cases} \nu x_1=p_{12}\,p_{43}'+p_{31}\,p_{42}'+p_{41}\,p_{23}', \\ \nu x_2=-p_{23}\,p_{42}'+p_{42}\,p_{23}', \\ \nu x_3=-p_{23}\,p_{43}'+p_{43}\,p_{23}', \\ \nu x_4=p_{42}\,p_{43}'-p_{43}\,p_{42}', \end{cases}$$

whence by (33) we have

$$(34)\quad \begin{cases} \nu x_1=p_{12}\begin{vmatrix} P_1 & P_2 \\ P_1' & P_2' \end{vmatrix}+p_{31}\begin{vmatrix} P_3 & P_1 \\ P_3' & P_1' \end{vmatrix}+p_{41}\begin{vmatrix} P_4 & P_1 \\ P_4' & P_1' \end{vmatrix}\equiv F_1, \\ \nu x_2=p_{23}\begin{vmatrix} P_1 & P_3 \\ P_1' & P_3' \end{vmatrix}+p_{42}\begin{vmatrix} P_4 & P_1 \\ P_4' & P_1' \end{vmatrix}\equiv F_2, \\ \nu x_3=p_{23}\begin{vmatrix} P_2 & P_1 \\ P_2' & P_1' \end{vmatrix}+p_{43}\begin{vmatrix} P_4 & P_1 \\ P_4' & P_1' \end{vmatrix}\equiv F_3, \\ \nu x_4=p_{42}\begin{vmatrix} P_1 & P_2 \\ P_1' & P_2' \end{vmatrix}+p_{43}\begin{vmatrix} P_1 & P_3 \\ P_1' & P_3' \end{vmatrix}\equiv F_4. \end{cases}$$

Similarly we can derive other three systems of equivalent expressions:

$$(34')\quad \begin{cases} \nu' x_1=p_{31}\begin{vmatrix} P_3 & P_2 \\ P_3' & P_2' \end{vmatrix}+p_{41}\begin{vmatrix} P_4 & P_2 \\ P_4' & P_2' \end{vmatrix}\equiv F_1', \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots; \end{cases}$$

$$(34'') \quad \begin{cases} \nu'' x_1 = p_{12} \begin{vmatrix} P_3 & P_2 \\ P_3' & P_2' \end{vmatrix} + p_{41} \begin{vmatrix} P_4 & P_3 \\ P_4' & P_3' \end{vmatrix} & \equiv F_1'' , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots ; \end{cases}$$

$$(34''') \quad \begin{cases} \nu''' x_1 = p_{12} \begin{vmatrix} P_4 & P_2 \\ P_4' & P_2' \end{vmatrix} + p_{31} \begin{vmatrix} P_3 & P_4 \\ P_3' & P_4' \end{vmatrix} & \equiv F_1''' , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots . \end{cases}$$

If the line (p) be not contained in the plane E_p, the point (x) corresponding to (p) is given by

$$(6) \quad \begin{cases} \rho x_1 = & -P_2 p_{12} + P_3 p_{31} + P_4 p_{41} \equiv \phi_1 , \\ \rho x_2 = P_1 p_{12} & - P_3 p_{23} + P_4 p_{42} \equiv \phi_2 , \\ \rho x_3 = -P_1 p_{31} + P_2 p_{23} & + P_4 p_{43} \equiv \phi_3 , \\ \rho x_4 = -P_1 p_{41} - P_2 p_{42} - P_3 p_{43} & \equiv \phi_4 . \end{cases}$$

Similarly if the line (p) be not contained in the plane E'_p, we have

$$(35) \quad \begin{cases} \rho' x_1 = & -P_2' p_{12} + P_3' p_{31} + P_4' p_{41} \equiv \phi_1' , \\ \rho' x_2 = P_1' p_{12} & - P_3' p_{23} + P_4' p_{42} \equiv \phi_2' , \\ \rho' x_3 = -P_1' p_{31} + P_2' p_{23} & + P_4' p_{43} \equiv \phi_3' , \\ \sigma' x_4 = -P_1' p_{41} - P_2' p_{42} - P_3' p_{43} & \equiv \phi_4' . \end{cases}$$

If (p) be contained neither in E_p nor in E'_p, (6) and (35) are equivalent to each other, which is self-evident geometrically. To prove this analytically, take the determinant

$$-p_{43}\, \Gamma \equiv \begin{vmatrix} P_1 & P_2 & P_3 & P_4 \\ P'_1 & P'_2 & P'_3 & P'_4 \\ 0 & -p_{43} & p_{42} & p_{23} \\ p_{43} & 0 & -p_{41} & p_{31} \end{vmatrix} .$$

Since Γ vanishes, by the well known theorem concerning the minors, we have

$$\frac{\phi_1}{\phi_1'} = \frac{\phi_2}{\phi_2'} = \frac{\phi_3}{\phi_3'} = \frac{\phi_4}{\phi_4'} .$$

20. Now the point corresponding to a given line (p) becomes indeterminate when and only when (p) satisfies

(36) $\qquad \Gamma = 0, \quad F_1 = F_2 = F_3 = F_4 = 0 ;$

or $\qquad \Gamma = 0, \quad F_1' = F_2' = F_3' = F_4' = 0 ;$

or $\qquad \Gamma = 0, \quad F_1'' = F_2'' = F_3'' = F_4'' = 0 ;$

or $$\Gamma=0, \quad F_1'''=F_2'''=F_3'''=F_4'''=0.$$

The totality of all these lines will be called *the fundamental figure F_p in the space Σ_p.*

Since we have the identities

$$F_i=P_1'\,\phi_i-P_1\,\phi_i', \quad F_i'=P_2'\,\phi_i-P_2\,\phi_i',$$
$$F_i''=P_3'\,\phi_i-P_3\,\phi_i', \quad F_i'''=P_4'\,\phi_i-P_4\,\phi_i',$$

the fundamental figure is given by

$$\Gamma=0,\ \phi_1=\phi_2=\phi_3=\phi_4=0,\ \phi_1'=\phi_2'=\phi_3'=\phi_4=0.$$

But by the identities

$$\begin{aligned}\Gamma&=P_1'\,\phi_1+P_2'\,\phi_2+P_3'\,\phi_3+P_4'\,\phi_4\\&=-(P_1\,\phi_1'+P_2\,\phi_2'+P_3\,\phi_3'+P_4\,\phi_4'),\end{aligned}$$

we arrive at the theorem:

The fundamental figure F_p may be defined by

(37) $$\phi_1=\phi_2=\phi_3=\phi_4=0,\ \phi_1'=\phi_2'=\phi_3'=\phi_4'=0;$$

so that it consists of all the lines which are common to the two singular figures of the two given principal coincidences (30).

Any line of the cubic complex Γ which does not belong to F_p corresponds to one point; any line of the fundamental figure to a range of points, the base being the given line. *The totality T_p of all the points corresponding to all lines (p) of the cubic complex Γ forms, in general, the whole space.*

The space Σ_x.

21. When any point (x) is given, the corresponding line (p) is, in general, determined uniquely. This line is the intersection of the two planes E_x and E_x':

(37) $$\begin{aligned}E_x:&\quad y_1U_1+y_2U_2+y_3U_3+y_4U_4=0,\\E_x':&\quad y_1U_1'+y_2U_2'+y_3U_3'+y_4U_4'=0,\end{aligned}$$

where we have put

(38) $$\left\{\begin{aligned}U_1'&= \qquad\quad -A_{12}'\,x_2+A_{31}'\,x_3+A_{41}'\,x_4\,,\\U_2'&=A_{12}'\,x_1 \qquad\quad -A_{23}'\,x_3+A_{42}'\,x_4\,,\\U_3'&=-A_{31}'\,x_1+A_{23}'\,x_2 \qquad\quad +A_{43}'\,x_4\,,\\U_4'&=-A_{41}'\,x_1-A_{42}'\,x_2-A_{43}'\,x_3 \qquad\quad ;\end{aligned}\right.$$

(39) $\quad A_{ik}'=a_{ik}'\,x_1+b_{ik}'\,x_2+c_{ik}'\,x_3+d_{ik}'\,x_4\,,\ (i,k=1,2,3,4\,;\ i\neq k).$

Hence we obtain

$$
(40)\quad \left\{
\begin{aligned}
&\rho p_{12}=\begin{vmatrix} U_4 & U_3 \\ U_4' & U_3' \end{vmatrix}\equiv\psi_{12}\,,\quad \rho p_{23}=\begin{vmatrix} U_4 & U_1 \\ U_4' & U_1' \end{vmatrix}\equiv\psi_{23}\,,\quad \rho p_{31}=\begin{vmatrix} U_4 & U_2 \\ U_4' & U_2' \end{vmatrix}\equiv\psi_{31}\,,\\
&\rho p_{41}=\begin{vmatrix} U_2 & U_3 \\ U_2' & U_3' \end{vmatrix}\equiv\psi_{41}\,,\quad \rho p_{42}=\begin{vmatrix} U_3 & U_1 \\ U_3' & U_1' \end{vmatrix}\equiv\psi_{42}\,,\quad \rho p_{43}=\begin{vmatrix} U_1 & U_2 \\ U_1' & U_2' \end{vmatrix}\equiv\psi_{43}\,.
\end{aligned}
\right.
$$

The totality T_x of all the lines corresponding to all points in Σ_x is obtained by eliminating x_1, x_2, x_3, x_4 from equations (30), (2), (3); so that *T_x consists of the cubic complex Γ.*

22. The line (p) corresponding to the point (x) becomes indeterminate, when and only when the two planes E_x and E_x' coincide. The totality of these points will be called *the fundamental figure F_x in the space Σ_x*.

Now the three quartic surfaces $\psi_{41}=\psi_{42}=\psi_{43}=0$ have a curve of degree 12, K^{12}, in common. For, the whole intersection K^{16} of

$$\frac{U_2}{U_2'}=\frac{U_3}{U_3'} \quad\text{and}\quad \frac{U_3}{U_3'}=\frac{U_1}{U_1'}\ (^1)$$

consists of the quartic curve $U_3=0$, $U_3'=0$ and K^{12}; and for K^{12} we have

$$\frac{U_1}{U_1'}=\frac{U_2}{U_2'}.$$

But since we have the identities

$$x_1\psi_{42}-x_2\psi_{41}\equiv x_4\psi_{12}\,,$$

---

(1) In order to prove that this intersection is in general a curve of degree 16, it is sufficient to show that the intersection has 16 points in the plane $x_1=0$. If we put, for example,

$$a_{41}=b_{41}=c_{41}=d_{41}=0,\quad a_{41}'=b_{41}'=c_{41}'=d_{41}'=0,$$
$$a_{42}=b_{42}=c_{42}=d_{42}=0,\quad a_{42}'=b_{42}'=c_{42}'=d_{42}'=0,$$

we have the 16 points:

$$x_2=0,\ x_3=0\,;\ x_2=0,\ x_4=0\,;\ \frac{x_2}{x_4}=-\frac{[A_{43}]}{[A_{32}]}=\frac{[A_{43}']}{[A_{23}']}\ \text{(3 points)}\,;$$

$$x_3=0,\ \frac{x_2}{x_4}=\frac{[A_{43}]\,[A_{12}']-[A_{12}]\,[A_{43}']}{[A_{12}]\,[A_{23}']-[A_{23}]\,[A_{12}']}\ \text{(3 points)}\,;$$

$$x_4=0,\ \frac{[A_{23}']}{[A_{23}]}=\frac{[A_{12}']x_2+[A_{31}']x_3}{[A_{12}]x_2+[A_{31}]x_3}\ \text{(3 points)}\,;$$

$$\frac{[A_{23}']}{[A_{23}]}=\frac{[A_{43}']}{[A_{43}]}=\frac{[A_{12}']x_2+[A_{31}']x_3}{[A_{12}]x_2+[A_{31}]x_3}\ \text{(5 points)}\,;$$

$[A_{ik}]$, $[A_{ik}']$ standing for $[A_{ik}]_{x_1=0}$, $[A_{ik}']_{x_1=0}$ respectively.

$$x_2\psi_{43} - x\psi_{342} \equiv x_4\psi_{23},$$
$$x_3\psi_{41} - x_1\psi_{43} \equiv x_4\psi_{31},$$

for the curve K^{12} it must be either

$x_4=0$, (one of $\psi_{12}, \psi_{23}, \psi_{31}$, at least, is not zero);

or $\psi_{12}=\psi_{23}=\psi_{31}=0, \quad (x_4 \neq 0)$;

or $x_4=0, \; \psi_{12}=\psi_{23}=\psi_{31}=0.$

Now the intersection of $\psi_{41}=0$ and $x_4=0$ consists of the line $x_1=0$ and the plane cubic K^3

$$(41) \qquad \begin{vmatrix} x_1 & x_2 & x_3 \\ [A_{23}]_{x_4=0} & [A_{31}]_{x_4=0} & [A_{12}]_{x_4=0} \\ [A_{23}']_{x_4=0} & [A_{31}']_{x_4=0} & [A_{12}']_{x_4=0} \end{vmatrix} = 0.$$

Similarly the intersection of $\psi_{42}=0$ and $x_4=0$ consists of $x_2=0$ and K^3; also that of $\psi_{43}=0$ and $x_4=0$ consists of $x_3=0$ and K^3. Hence K^{12} breaks up into K^3 in the plane $x_4=0$ and another curve K^9; and it is easily seen that K^3 does not belong to $\psi_{12}=\psi_{23}=\psi_{31}=0$. Therefore for K^9 we must have

$$\psi_{12}=\psi_{23}=\psi_{31}=0;$$

in other words, the six quartic surfaces

$$(42) \qquad \psi_{12}=0, \; \psi_{23}=0, \; \psi_{31}=0, \; \psi_{41}=0, \; \psi_{42}=0, \; \psi_{43}=0$$

have K^9 (the fundamental figure F_x) in common. Consequently we arrive at the theorem:

The fundamental figure F_x of space Σ_x is the space curve of degree 9. Any point of the fundamental figure corresponds to a flat pencil whose vertex is the given point.

23. In § 22 we have seen that any point (x) of the cubic (41) in the plane $x_4=0$ corresponds to a definite line (p) in that plane; and the converse is also true.

More generally we can prove the theorem: *In order that the line (p) corresponding to a point (x) may be contained in any fixed plane E, it is necessary and sufficient that the point (x) should lie on a certain cubic curve in that plane E.*

For, by a suitable choice of collineation, the two principal coincidences (1, 1) in the plane E, which are derived from (30) and (2)–(5) (see § 5), take the forms

$$\begin{cases} u_1' x_1' + u_2 x_2' + u_3' x_3' = 0, \\ u_1' A_1 + u_2' A_2 + u_3' A_3 = 0, \\ u_1' A_1' + u_2' A_2' + u_3' A_3' = 0, \end{cases}$$

where x_1', x_2', x_3' and u_1', u_2', u_3' are the point and line coordinates in the plane E respectively; and A_1, A_1',······are linear forms of x_1', x_2', x_3'. Eliminating u_1', u_2', u_3' from these equations we have

$$\text{(43)} \qquad \begin{vmatrix} x_1' & x_2' & x_3' \\ A_1 & A_2 & A_3 \\ A_1' & A_2' & A_3' \end{vmatrix} = 0,$$

from which the theorem follows.

When the point (x') *describes the cubic (43), the corresponding line envelopes the curve of the third class, the reciprocal of (43) with respect to the conic*

$$x_1'^2 + x_2'^2 + x_3'^2 = 0.$$

One-to-one correspondence between the lines of the cubic complex Γ and the points of space.

24. It follows from § 19 and § 21 that *a (1, 1) correspondence has been established between the lines* (p) *of the cubic complex* Γ *and the points* (x) *of space by aid of the formulæ (34) and (40).*

This correspondence is involutory in general. For, let (p) be the line corresponding to any given point (x) and let (x') be the point corresponding to the line (p). Then

$$\Sigma P_i x_i = 0, \ \ \Sigma P_i' x_i = 0,$$

and

$$\Sigma P_i x_i' = 0, \ \ \Sigma P_i' x_i' = 0.$$

Hence for any two constants λ, μ we have

$$\Sigma P_i(\lambda x_i + \mu x_i') = 0, \ \ \Sigma P_i'(\lambda x_i + \mu x_i') = 0\,;$$

so that all the points of the line (p) which joins (x) and (x') correspond to the same line (p), which is not the case in general. Hence (x) and (x') must coincide. Similarly, if (x) be the point corresponding to any given line (p) and (p') the line corresponding to the point (x), then (p') coincides with (p).

The essential property of this correspondence is that the line corresponding to a given point passes through that point and the point corres-

ponding to a given line lies in that line. Prof. Noether's representations for the linear complex[1] and for the quadratic complex[2], and that of Prof. Weiler for the tetrahedral complex[3] do not possess this property.

25. Now we exclude the figure common to F_p' and F_x', and then denote by Σ_p' the line space and by Σ_x' the point space.

Let x, y, z be the rectangular coordinates in Σ_x', and let

$$\rho x_1 = x, \ \rho x_2 = y, \ \rho x_3 = z, \ \rho x_4 = 1.$$

A curve drawn from any origin, so that at every point (x) on it its tangent is the line (p) corresponding to that point (x), is called *a curve of the simultaneous principal coincidences.* By (40) these curves are determined as the integral curves of the system of differential equations

$$(44) \qquad \frac{dx}{\begin{vmatrix} U_2 & U_3 \\ U_2' & U_3' \end{vmatrix}} = \frac{dy}{\begin{vmatrix} U_3 & U_1 \\ U_3' & U_1' \end{vmatrix}} = \frac{dz}{\begin{vmatrix} U_1 & U_2 \\ U_1' & U_2' \end{vmatrix}}$$

so that they constitute a congruence of curves.

In general, there is one and only one integral curve which passes through a given point in Σ_x'; and there is one and only one integral curve which touches a given line of Γ in Σ_p'.

Suppose that a line is an integral curve. Since the line corresponds to all the points on it, it must belong to the fundamental figure F_p' in Σ_p'. The analytical condition that an integral curve should be a line is that the relations

$$(45) \qquad \frac{\dfrac{\partial Y}{\partial z} - \dfrac{\partial Z}{\partial y}}{X} = \frac{\dfrac{\partial Z}{\partial x} - \dfrac{\partial X}{\partial z}}{Y} = \frac{\dfrac{\partial X}{\partial y} - \dfrac{\partial Y}{\partial x}}{Z} \text{ [4]}$$

should satisfy for every point of that curve, where we have put

$$X = \frac{\begin{vmatrix} U_2 & U_3 \\ U_2' & U_3' \end{vmatrix}}{W}, \quad Y = \frac{\begin{vmatrix} U_3 & U_1 \\ U_3' & U_1' \end{vmatrix}}{W}, \quad Z = \frac{\begin{vmatrix} U_1 & U_2 \\ U_1' & U_2' \end{vmatrix}}{W},$$

$$W = \sqrt{\begin{vmatrix} U_2 & U_3 \\ U_2' & U_3' \end{vmatrix}^2 + \begin{vmatrix} U_3 & U_1 \\ U_3' & U_1' \end{vmatrix}^2 + \begin{vmatrix} U_1 & U_2 \\ U_1' & U_2' \end{vmatrix}^2}.$$

---

(1) See Jessop, Line complex, p. 187.

(2) Jessop, loc. cit., p. 179.

(3) Weiler, Zeitschr. f. Math. u. Physik, 22 (1877), p. 261.

(4) Goursat, Vorlesungen über d. part. Diff.-Gleichungen 1. Ordnung (1893) p. 41.

If the above relations (45) *be satisfied identically,* the integral curves form a line congruence. This congruence is the totality T_x' of lines corresponding to all points in Σ_x', and at the same time it is nothing but the fundamental figure F_p' in Σ_p'.

Similarly, in Σ_p' the point of contact of any tangent to any curve of the simultaneous principal coincidences is the point corresponding to that tangent. *If each of these curves reduce to a point,* the set of such points is the totality T_p' of points corresponding to all the lines in Σ_p' and at the same time it is nothing but the fundamental figure F_x' in Σ_x'.

26. If certain special connections exist between the constants a, b, c, d; a', b', c', d' in (30), the complex Γ is modified in character. Especially Γ may take the form

$$\Gamma=(p_{12}\,p_{43}+p_{23}\,p_{41}+p_{31}\,p_{42})P_0,$$

where P_0 is a linear form of p_{ik}. (For an example, see § 30.) In this case the cubic complex vanishes, and consequently to any given line in space corresponds a point and to any given point in space corresponds a line.

For the convenience of readers, a brief summary of some special varieties will be given in the following:

I. *Correspondence*

(i) Involutory. (General case § 24; Ex. 1 below.)
(ii) Non-involutory. (Ex. 2; Ex. 3; Ex. 4.)

II. T_p

We exclude all the points corresponding to F_p. Then
(i) ∞^3 points. (General case § 20; Ex. 1.)
(ii) ∞^2 points. (Ex. 4.)
(iii) ∞^1 points. (Ex. 2; Ex. 3.)

III. T_x

We exclude all the lines corresponding to F_x. Then
(i) ∞^3 lines. (General case § 21; Ex. 1.)
(ii) ∞^2 lines. (Ex. 2; Ex. 3; Ex. 4.)

IV. F_p

(i) Ruled surface. (General case § 20.)
(ii) Proper congruence. (Ex. 2; Ex. 3.)
(iii) Improper congruence. (Ex. 1; Ex. 4.)

V. F_x

(i) ∞^1 points. (General case § 22 ; Ex. 2.)
(ii) ∞^2 points. (Ex. 1 ; Ex. 3 , Ex. 4.)

VI. Γ

(i) Γ vanishes. (Ex. 4.)
(ii) Γ breaks up into two or three complexes. (Ex. 1 ; Ex. 3.)
(iii) Γ does not break. (General case ; Ex. 2.)

Some special varieties.

Example I([1]).

[I(i), II(i), III(i), IV(iii), V(ii), VI(ii)]

27. If we take

$$P_1 = a(p_{42}-p_{43}), \quad P_2 = b(p_{43}-p_{41}), \quad P_3 = c(p_{41}-p_{42}), \quad P_4 = 0;$$
$$P_1' = a^2(bp_{42}-cp_{43}), \quad P_2' = b^2(cp_{43}-ap_{41}), \quad P_3' = c^2(ap_{41}-bp_{42}), \quad P_4' = 0,$$

a, b, c are arbitrary constants, then the cubic complex Γ becomes

$$\{a(b-c)p_{41}+b(c-a)p_{42}+c(a-b)p_{43}\}(abp_{12}p_{43}+bcp_{23}p_{41}+cap_{31}p_{42})=0,$$

which breaks up into the linear complex L:

$$(46) \qquad L \equiv a(b-c)p_{41}+b(c-a)p_{42}+c(a-b)p_{43}=0$$

and the tetrahedral complex T:

$$(47) \qquad T \equiv abp_{12}p_{43}+bcp_{23}p_{41}+cap_{31}p_{42}=0.$$

Also we have

$$\psi_{12} = -(a-b)x_1x_2\psi_0, \quad \psi_{23} = -(b-c)x_2x_3\psi_0, \quad \psi_{31} = -(c-a)x_3x_1\psi_0,$$
$$\psi_{41} = ax_1x_4\psi_0, \quad \psi_{42} = bx_2x_4\psi_0, \quad \psi_{43} = cx_3x_4\psi_0,$$

where

$$\psi_0 \equiv x_4[a^2(b-c)x_1+b^2(c-a)x_2+c^2(a-b)x_3]:$$

so that the fundamental figure F_x consists of the two planes given by $\psi_0=0$ and the four vertices of the tetrahedron of reference $A_1A_2A_3A_4$:

$$x_2=x_3=x_4=0, \quad x_3=x_4=x_1=0, \quad x_4=x_1=x_2=0, \quad x_1=x_2=x_3=0.$$

If a point (x) be not contained in $\psi_0=0$, the line (p) corresponding to that point is given by

$$(48) \quad \begin{cases} \rho p_{12} = -(a-b)x_1x_2, & \rho p_{23} = -(b-c)x_2x_3, & \rho p_{31} = -(c-a)x_3x_1, \\ \rho p_{41} = ax_1x_4, & \rho p_{42} = bx_2x_4, & \rho p_{43} = cx_3x_4. \end{cases}$$

([1]) For the dynamical meaning of this example, see Ogura, loc. cit..

This line belongs to the tetrahedral complex T, but not to the linear complex L; for

$$L=\rho\psi_0\neq 0.$$

Hence *if we exclude the common factor* ψ_0 (being absorbed into the proportional factor ρ), *it is sufficient to consider the tetrahedral complex T only.*

Next since

$$\begin{aligned}
\phi_1 &=bp_{12}(p_{41}-p_{43})+cp_{31}(p_{41}-p_{42}),\\
\phi_2 &=cp_{23}(p_{42}-p_{41})+ap_{12}(p_{42}-p_{43}),\\
\phi_3 &=ap_{31}(p_{43}-p_{42})+bp_{23}(p_{43}-p_{41}),\\
\phi_4 &=(b-a)p_{41}p_{42}+(c-b)p_{42}p_{43}+(a-c)p_{43}p_{41};\\
\phi_1' &=b^2p_{12}(ap_{41}-cp_{43})+c^2p_{31}(ap_{41}-bp_{42}),\\
\phi_2' &=c^2p_{23}(bp_{42}-ap_{41})+a^2p_{12}(bp_{42}-cp_{43}),\\
\phi_3' &=a^2p_{31}(cp_{43}-bp_{42})+b^2p_{23}(cp_{43}-ap_{41}),\\
\phi_4' &=(b-a)abp_{41}p_{42}+(c-b)bcp_{42}p_{43}+(a-c)cap_{43}p_{41},
\end{aligned}$$

the point corresponding to a line (p) belonging to the tetrahedral complex T is given by

(49) $\lambda x_1=bcp_{23}p_{41}$, $\lambda x_2=cap_{23}p_{42}$, $\lambda x_3=abp_{23}p_{43}$, $\lambda x_4=a(c-b)p_{42}p_{43}$;

which may be written

(49′) $\lambda' x_1=bcp_{31}p_{41}$, $\lambda' x_2=cap_{31}p_{42}$, $\lambda' x_3=abp_{31}p_{43}$, $\lambda' x_4=b(a-c)p_{43}p_{41}$;

or

(49″) $\lambda'' x_1=bcp_{12}p_{41}$, $\lambda'' x_2=cap_{12}p_{42}$, $\lambda'' x_3=abp_{12}p_{43}$, $\lambda'' x_4=c(b-a)p_{41}p_{42}$.

It follows that the fundamental figure F_p in Σ_p consists of all the lines passing through every vertex of the tetrahedron of reference $A_1A_2A_3A_4$ and of all lines contained in every face of the tetrahedron.

Thus *we have established a (1, 1) correspondence between the lines of the tetrahedral complex T and the points of space. This correspondence is involutory:* for, let (p) be the line corresponding to a given point (x), and (y) be the point corresponding to the line (p). Then

$$\lambda y_1=\mu x_1, \qquad \lambda y_2=\mu x_2, \qquad \lambda y_3=\mu x_3, \qquad \lambda y_4=\mu x_4,$$

where

$$\mu=abc(c-b)x_2x_3x_4\rho^{-2}.$$

Similarly if (x) be the point corresponding to a given line (p), and (q) be the line corresponding to the point (x), then we can see that

$$\rho q_{ik}=\sigma p_{ik}, \qquad \sigma=a^2bc(c-b)p_{23}p_{42}p_{43}\lambda^{-2},$$

by aid of the equations

$$p_{12}p_{43}+p_{23}p_{41}+p_{31}p_{42}=0, \qquad abp_{12}p_{43}+bcp_{23}p_{41}+cap_{31}p_{42}=0.$$

Now in order to interpret the meaning of this correspondence, take the rectangular coordinates x, y, z and put

$$\rho x_1=x, \qquad \rho x_2=y, \qquad \rho x_3=z, \qquad \rho x_4=1.$$

Consider a congruence of W-curves (the curves of simultaneous principal coincidences) defined by

$$x=Ae^{at}, \qquad y=Be^{bt}, \qquad z=Ce^{ct},$$

where t denotes the parameter and A, B, C are arbitrary constants. It is well known that there is always one and only one curve of the system which passes through a given point (excluding the varieties of the tetrahedron of reference $A_1A_2A_3A_4$). Since

$$\frac{dx}{dt}=ax, \qquad \frac{dy}{dt}=by, \qquad \frac{dz}{dt}=cz,$$

the tangent (p) *to the* W*-curve at the point* (x) *is*

$$\begin{aligned} &\rho p_{12}=(b-a)xy, \quad \rho p_{23}=(c-b)yz, \quad \rho p_{31}=(a-c)zx, \\ &\rho p_{41}=ax, \qquad\qquad \rho p_{42}=by, \qquad\qquad \rho p_{43}=cz; \end{aligned} \tag{48}$$

which is nothing but the line corresponding to the given point (x).

Next, it is well known that there is always one and only one W-curve of the system which touches a given line of the tetrahedral complex T (excluding the fundamental figure F_p). But since the correspondence is involutory, *the point corresponding to the given line* (p) *is the point of contact of this line and the* W*-curve.*

These results hold good even when the coordinates x_1, x_2, x_3, x_4 are taken in general, if we adopt the system of W-curves

$$x_1=C_1e^{a_1t}, \qquad x_2=C_2e^{a_2t}. \qquad x_3=C_3e^{a_3t}, \qquad x_4=C_4e^{a_4t},$$

where

$$a_1-a_4=a, \qquad a_2-a_4=b, \qquad a_3-a_4=c,$$

and C_1, C_2, C_3, C_4 are constants.

The totality T_p of all the points corresponding to all the lines in the tetrahedral complex T (excluding the fundamental figure F_p) consists of point space, and the totality T_x of all the lines corresponding to all the points in space (excluding the fundamental figure F_x) consists of the tetrahedral complex T(1).

(1) All the tangents, passing through a given point, to all the W-curves of the system form a quadric cone; and *the locus of points of contact is a space cubic passing through the given point.* All the tangents lying in a given plane, to all the curves of the system, envelopes a conic; and *the locus of points of contact is a line touching the cubic.*

Thus we have had a quadratic duality of the points in space and the lines in the tetrahedral complex, and a system of space W-curves as the curves of the simultaneous principal coincidences. This example may be looked upon as the natural extension of the principal coincidence (1, 1) in a plane. (See the introduction of this paper.)

EXAMPLE II.

[I(ii), II(iii), III(ii), IV(ii), V(i), VI(iii)]

28. If we put

$$P_1 = -p_{43}, \qquad P_2 = p_{42}, \qquad P_3 = p_{23}, \qquad P_4 = 0;$$
$$P_1' = 0, \qquad P_2' = p_{23}, \qquad P_3' = p_{31}, \qquad P_4' = p_{12},$$

then the cubic complex Γ becomes

$$p_{12}\,p_{41}\,p_{43} + p_{31}^2\,p_{43} - p_{12}\,p_{42}^2 + p_{23}p_{31}\,p_{42} - p_{23}^3 - 2p_{12}p_{23}p_{43} = 0,$$

which is nothing but the totality of all lines cutting the space cubic

$$(50) \qquad \rho x_1 = \theta^3, \qquad \rho x_2 = \theta^2, \qquad \rho x_3 = \theta, \qquad \rho x_4 = 1,$$

θ being the parameter(1).

Since equations (6) and (35) become

$$\begin{aligned}
\nu x_1 &= \phi_1 = -p_{42}p_{12} + p_{23}p_{31} = \lambda\theta^3,\\
\nu x_2 &= \phi_2 = -p_{42}p_{12} - p_{23}^2 \quad = \lambda\theta^2,\\
\nu x_3 &= \phi_3 = \quad p_{43}p_{31} + p_{42}p_{23} \;\; = \lambda\theta,\\
\nu x_4 &= \phi_4 = \quad p_{41}p_{43} - p_{42}^2 - p_{23}p_{43} = \lambda;
\end{aligned}$$

and

$$\begin{aligned}
\nu' x_1 &= \phi_1' = -p_{23}p_{12} + p_{31}^2 + p_{12}p_{41} = \lambda'\theta^4,\\
\nu' x_2 &= \phi_2' = -p_{31}p_{23} + p_{12}p_{42} \qquad = \lambda'\theta^3,\\
\nu' x_3 &= \phi_3' = \quad p_{23}^2 + p_{12}p_{43} \quad = \lambda'\theta^2,\\
\nu' x_4 &= \phi_4' = -p_{23}p_{42} - p_{31}p_{43} = \lambda'\theta.
\end{aligned}$$

respectively, the point corresponding to a given line (p) in Γ coincides with *the point of intersection of the line with the space cubic (50)* (2). Hence the fundamental figure F_p consists of all the chords of the cubic, which form a congruence of the first order and the third class(3); the totality T_p of all the points corresponding to all lines of Γ consists of the space cubic (50).

---

(1) Clebsch-Lindemann, loc. cit., II, 1, p. 244; Wood, The twisted cubic (1913), p. 17.

(2) Wood, loc. cit., pp. 17–18.

(3) Reye, Geometrie der Lage, II, 4. Aufl. (1907), p. 162.

Next the line (p) corresponding to a point (x) in Σ_x is

$$(51)\quad \begin{cases} \mu p_{41}=\psi_{41}\equiv(x_1x_3-x_2^2)(x_2x_4-x_3^2)+(x_1x_4-x_2x_3)^2, \\ \mu p_{42}=\psi_{42}\equiv(x_2x_4-x_3^2)(x_1x_4-x_2x_3), \\ \mu p_{43}=\psi_{43}\equiv(x_2x_4-x_3^2)^2, \\ \mu p_{12}=\psi_{12}\equiv-(x_1x_3-x_2^2)^2, \\ \mu p_{23}=\psi_{23}\equiv-(x_1x_3-x_2^2)(x_2x_4-x_3^2), \\ \mu p_{31}=\psi_{31}\equiv(x_1x_3-x_2^2)(x_1x_4-x_2x_3); \end{cases}$$

so that the fundamental figure F_x:

$$x_1x_3-x_2^2=0, \qquad x_1x_4-x_2x_3=0, \qquad x_2x_4-x_3^2=0$$

is the cubic (50).

Now we prove the theorem: *The line* (p) *corresponding to* (x) *is the unique chord of the cubic through the given point* (x). Since this chord passes through the two points θ_1, θ_2 which satisfy the quadratic equation

$$(x_2x_4-x_3^2)\theta^2-(x_1x_4-x_2x_3)\theta+(x_1x_3-x_2^2)=0\text{ [1]},$$

the line coordinates $(\bar{p}_{ik})$ of this chord are

$$\begin{aligned} &\nu\bar{p}_{12}=\theta_1^2\theta_2^2(\theta_1-\theta_2), && \nu\bar{p}_{23}=\theta_1\theta_2(\theta_1-\theta_2), \\ &\nu\bar{p}_{31}=-\theta_1\theta_2(\theta_1^2-\theta_2^2), && \nu\bar{p}_{41}=[\theta_1\theta_2-(\theta_1+\theta_2)^2](\theta_1-\theta_2), \\ &\nu\bar{p}_{42}=-(\theta_1^2-\theta_2^2), && \nu\bar{p}_{43}=-(\theta_1-\theta_2), \end{aligned}$$

where

$$\theta_1+\theta_2=\frac{x_1x_4-x_2x_3}{x_2x_4-x_3^2}, \qquad \theta_1\theta_2=\frac{x_1x_4-x_2^2}{x_2x_4-x_3^2};$$

so that this chord coincides with the line (51).

The totality T_x' of all the lines corresponding to all the points in Σ_x consists of the congruence of chords of the cubic (50).

In this case T_p coincides with F_x and T_x with F_p. (See §25.)

Example III.

[I(ii), II(iii), III(ii), IV(ii), V(ii), VI(ii)]

29. If we take

$$\begin{aligned} &P_1=p_{12}, && P_2=p_{12}, && P_3=0, && P_4=0, \\ &P_1'=0, && P_2'=0, && P_3'=p_{43}, && P_4'=p_{43}, \end{aligned}$$

we have

$$\Gamma\equiv p_{12}p_{43}(p_{23}-p_{31}-p_{41}-p_{42}).$$

The complex $\Gamma=0$ consists of (i) all lines cutting the line A_1A_2 $(x_3=x_4=0)$, (ii) all lines cutting A_3A_4 $(x_1=x_2=0)$ and (iii) all lines

(1) Wood, loc. cit., p. 11.

cutting the line which joins the two points $P(1,-1,0,0)$ and $Q(0,0,1,-1)$.

Since

$$\phi_1=-p_{12}^2,\quad \phi_2=p_{12}^2,\quad \phi_3=p_{12}(p_{23}-p_{31}),\ \phi_4=-p_{12}(p_{41}+p_{42});$$
$$\phi_1'=p_{43}(p_{31}+p_{41}),\ \phi_2'=p_{43}(-p_{23}+p_{42}),\ \phi_3'=p_{43}^2,\quad \phi_4'=-p_{43}^2,$$

the fundamental figure F_p is the linear congruence

$$p_{12}=0,\qquad p_{43}=0.$$

Any given line (p) of the complex (excluding F_p) corresponds to the point of intersection of this line and one of the lines A_1A_2, A_3A_4, PQ. The totality T_p of the points corresponding to all the lines (excluding F_p) consists of the lines A_1A_2, A_3A_3 and PQ.

Next since

$$\psi_{12}=0,\quad \psi_{23}=-(x_1+x_2)(x_3+x_4)x_2x_3,\quad \psi_{31}=(x_1+x_2)(x_3+x_4)x_3x_1,$$
$$\psi_{41}=(x_1+x_2)(x_3+x_4)x_4x_1,\quad \psi_{42}=(x_1+x_2)(x_3+x_4)x_4x_2,\ \psi_{43}=0,$$

the fundamental figure F_x consists of the two planes A_1A_2Q, A_3A_4P and the two lines A_1A_2, A_3A_4. Any given point (x) corresponds to the line cutting A_1A_2, A_3A_4 and passing through that given point. The totality T_x of the lines corresponding to all the points (excluding F_x) consists of the linear congruence

$$p_{12}=0,\qquad p_{43}=0.$$

In this example T_p coincides with F_x' (the two lines A_1A_2, A_3A_4) and T_x' with F_p' (lines of the linear congruence excluding the two flat pencils $Q(A_1A_2)$ and $P(A_3A_4)$). (See §25.)

Example IV.

[I(ii), II(ii), III(ii), IV(iii), V(ii), VI(i)]

30. If we take

$$\begin{cases} P_1= & \quad ap_{42}-cp_{43}, \\ P_2=-ap_{41} & \quad +bp_{43}, \\ P_3= cp_{41}-bp_{42} & , \\ P_4= dp_{41}+ep_{42}+fp_{43}, \end{cases}\qquad \begin{cases} P_1'= & \quad a'p_{42}-c'p_{43}, \\ P_2'=-a'p_{41} & \quad +b'p_{43}, \\ P_3'= c'p_{41}-b'p_{42} & , \\ P_4'= d'p_{41}+e'p_{42}+f'p_{43}, \end{cases}$$

then

$$\Gamma\equiv[(ac'-a'c)p_{41}+(ba'-b'a)p_{42}+(cb'-c'b)p_{43}](p_{12}p_{43}+p_{23}p_{41}+p_{31}p_{42});$$

so that *the cubic complex Γ vanishes.* Therefore we may take any line in space.

As we have seen in § 7,

$$\phi_1 = p_{41}\phi_0, \qquad \phi_2 = p_{42}\phi_0, \qquad \phi_3 = p_{43}\phi_0, \qquad \phi_4 = 0,$$
$$\phi_0 \equiv a p_{12} + b p_{23} + c p_{31} + d p_{41} + e p_{42} + f p_{43};$$
$$\phi_1' = p_{41}\phi_0', \qquad \phi_2' = p_{42}\phi_0' \qquad \phi_3' = p_{43}\phi_0', \qquad \phi_4' = 0,$$
$$\phi_0' \equiv a' p_{12} + b' p_{23} + c' p_{31} + d' p_{41} + e' p_{42} + f' p_{43};$$

whence the fundamental figure F_p consists of the linear congruence

$$\Phi_0 = \Phi_0' = 0$$

and

$$p_{41} = p_{42} = p_{43} = 0,$$

which is the totality of all the lines contained in the plane $x_4 = 0$.

Any given line (p) corresponds to the point

$$\rho x_1 = p_{41}, \qquad \rho x_2 = p_{42}, \qquad \rho x_3 = p_{43}, \qquad \rho x_4 = 0,$$

which is nothing but the point of intersection of (p) and the plane $x_4 = 0$. The totality T_p of the points corresponding to all the lines in space (excluding F_p) form the plane $x_4 = 0$.

Next we have

$$\begin{aligned}
U_1 &= x_4(\qquad\quad - a x_2 + c x_3 + d x_4),\\
U_2 &= x_4(\; a x_1 \qquad\quad - b x_3 + e x_4),\\
U_3 &= x_4(-c x_1 + b x_2 \qquad\quad + f x_4),\\
U_4 &= x_4(-d x_1 - e x_2 - f x_3 \qquad\quad);\\
U_1' &= x_4(\qquad\quad - a' x_2 + c' x_3 + d' x_4),\\
&\ldots\ldots\ldots\ldots\ldots\ldots;
\end{aligned}$$

so that the fundamental figure F_x consists of the double plane

$$x_4^2 = 0$$

and the two directrices of the linear congruence

$$\phi_0 = \phi_0' = 0.$$

Any given point corresponds to the line which belongs to the linear congruence and passes through the given point. The totality T_x of the lines corresponding to all the points in space (excluding F_x) is the linear congruence.

In this example F_p' coincides with T_x' (the linear congruence), and F_x' with T_p' (the plane $x_4 = 0$).

Theorems on Convergent Integrals,

by

TETSUZÔ KOJIMA, Sendai.

Throughout the present paper, let $u(x)$ be a bounded and integrable function in the interval $a \leqq x \leqq x_1$, x_1 being arbitrary, and $a(x, y)$ a function defined in the domain $a < x$, $a \leqq y \leqq x$, in such a way that 1° it is integrable in y for each x, $a \leqq y \leqq x$; 2° for a given arbitrary number $q\,(> a)$ there corresponds a number h such that, for all $x \geqq h$ the function $a(x, y)$ is uniformly continuous in y, $a \leqq y \leqq q$. We shall study in the following lines the necessary and sufficient condition that

$$\lim_{x \to \infty} \int_a^x a(x, y)\, u(y)\, dy$$

exists, whenever 1° $\lim\limits_{x \to \infty} u(x)$ exists; 2° $u(x)$ is bounded in $a \leqq x$; 3° $u(x)$ is monotonic and $\lim\limits_{x \to \infty} u(x)$ exists.

I.

Theorem I. The necessary and sufficient conditions that

$$\lim_{x \to \infty} \int_a^x a(x, y)\, u(y)\, dy$$

exists(1), *whenever* $\lim\limits_{x \to \infty} u(x)$ *exists, are that:*

i. $$\lim_{x \to \infty} \int_a^x a(x, y)\, dy \quad \textit{exists,}$$

ii. $$\lim_{x \to \infty} a(x, y) \quad \textit{exists,} \quad y \geqq a,$$

iii. $$\int_a^x |\, a(x, y)\, |\, dy < M,$$

where M is a positive constant.

When these conditions are fulfilled, we have

---

(1) Throughout this paper the words "the limit exists" mean that the limit exists and is finite.

$$\lim_{x\to\infty}\int_a^x a(x,y)\,u(y)\,dy=\lim_{x\to\infty}u(x)\int_a^x a(x,y)\,dy+\int_a^\infty b(y)\{u(y)-\lim_{x\to\infty}u(x)\}dy,$$

where

$$b(y)=\lim_{x\to\infty}a(x,y).$$

As the theorem can be proved in a similar manner as in Mr. Silverman's paper, "On the notion of summability for the limit of a function of continuous variable," Trans. Amer. Math. Soc., 17, 1916, provided that the following lemmas 1, 2 (a) and 3 be taken into consideration, we shall omit the proof.

Lemma 1. If $\lim_{x\to\infty} a(x,y)$ *exists, then* $b(y)=\lim_{x\to\infty}a(x,y)$ *is continuous in* y *in the interval* $a\leqq y\leqq q$, q *being arbitrary and the limit is approached uniformly with respect to* y *in the same interval.*

Let η be a value of y in the interval $a\leqq y\leqq q$. Since $a(x,y)$ is continuous in y uniformly with respect to x, we have

$$|a(x,\eta)-a(x,y)|<\frac{\varepsilon}{4},\ |\eta-y|<\delta,\ x\geqq h,$$

where ε is a given arbitrary small positive constant, and δ depends only upon ε and h. From this inequality together with

$$|a(x,\eta)-b(\eta)|<\frac{\varepsilon}{4},\quad x>h',$$

which results from hypothesis, it follows that

$$(1)\qquad |a(x,y)-b(\eta)|<\frac{\varepsilon}{2},\quad |\eta-y|<\delta,\quad x\geqq h_1,$$

where h_1 is the greater of h and h'; hence we have

$$(2)\qquad |b(y)-b(\eta)|\leqq\frac{\varepsilon}{2},\quad |\eta-y|<\delta,$$

so that the function $b(y)$ is continuous in y in the interval $a\leqq y\leqq q$. From the inequalities (1) and (2), we have

$$(3)\qquad |a(x,y)-b(y)|<\varepsilon,\quad |\eta-y|<\delta,\quad x\geqq h_1.$$

Thus, to any point η in the interval $a\leqq y\leqq q$, there corresponds a sub-interval for which the relation (3) holds; therefore, by Borel-Lebesgue's theorem, the interval $a\leqq y\leqq q$ may be covered by a finite number of the sub-intervals; accordingly we know that there exists a number k such that the relation

$$|a(x,y)-b(y)|<\varepsilon$$

holds for any $x > k$ and for any y in the interval $a \leqq y \leqq q$, namely

$$\lim_{x\to\infty} a(x,y)=b(y) \quad \text{uniformly in } y,\ a \leqq y \leqq q.$$

Lemma 2. If $\lim\limits_{x\to\infty} a(x,y)$ *exists, and* $\int_a^x |a(x,y)|\,dy < M$, *$M$ being a positive constant, then*

$$(a) \qquad \int_a^\infty |b(y)|\,dy \ \textit{converges},$$

and

$$(b) \qquad \int_a^\infty |b(y)|\,dy \leqq \varliminf_{x\to\infty} \int_a^x |a(x,y)|\,dy.$$

By Lemma 1, the function $b(y)$ is continuous, so that $|b(y)|$ is integrable in any finite interval. Write

$$a(x,y)-b(y)=c(x,y),$$

and let X be a number arbitrarily chosen, then

$$\int_a^x |a(x,y)|\,dy \geqq \int_a^X |a(x,y)|\,dy = \int_a^X |b(y)+c(x,y)|\,dy$$

$$\geqq \int_a^X |b(y)|\,dy - \int_a^X |c(x,y)|\,dy, \quad x > X.$$

Since by Lemma 1 $c(x,y)$ approaches to zero uniformly for y, $a \leqq y \leqq X$, when x tends to infinity, we have for a given arbitrary positive quantity ε

$$|c(x,y)| < \frac{\varepsilon}{2(X-a)}, \quad x > X'.$$

Hence we have

$$(4) \qquad \int_a^x |a(x,y)|\,dy > \int_a^X |b(y)|\,dy - \frac{\varepsilon}{2}, \quad x > \text{Max.}\ (X, X');$$

accordingly

$$\int_a^X |b(y)|\,dy < M + \frac{\varepsilon}{2},$$

so that

$$\int_a^\infty |b(y)|\,dy$$

converges. If we put

$$\int_a^\infty |b(y)|\,dy = B,$$

then there exists a number X such that the inequality

$$\int_a^X |b(y)|\,dy > B - \frac{\varepsilon}{2}$$

is satisfied, and for this number X the inequality (4) holds for any x greater than a fixed number, accordingly we have

$$\int_a^x |a(x, y)|\,dy > B - \varepsilon$$

for any x greater than a fixed number, so that

$$\varliminf_{x \to \infty} \int_a^x |a(x, y)|\,dy \geqq B.$$

Lemma 3. If a function $f(x)$ is bounded and integrable, $a \leqq x \leqq b$, then we can define a function $\varphi(x)$, which is integrable, changes its sign only a finite number of times in the interval, and satisfies the inequality

$$\int_a^b |f(x) - \varphi(x)|\,dx < \varepsilon,$$

where ε is a given arbitrary small positive quantity.

As $f(x)$ is integrable, we can determine $n+1$ points $x_0 = a$, $x_1, x_2, \cdots$ $\cdots$, x_{n-1}, $x_n = b$ in (a, b), satisfying the inequality

$$\sum_{i=1}^{n} (x_i - x_{i-1}) D_i < \varepsilon,$$

where D_i denotes the fluctuation of $f(x)$ in the closed sub-interval (x_{i-1}, x_i). Define a function $\varphi(x)$ such that

$$\varphi(x) = f\left(\frac{x_{i-1} + x_i}{2}\right), \quad x_{i-1} \leqq x < x_i, \quad i = 0, 1, 2, \cdots\cdots, n-1,$$

$$\varphi(x) = f\left(\frac{x_{n-1} + b}{2}\right), \quad x_{n-1} \leqq x \leqq b.$$

Then, $\varphi(x)$ changes its sign at most $(n-1)$ times, and is integrable, accordingly $|f(x) - \varphi(x)|$ is integrable. If we denote the upper limit of $|f(x) - \varphi(x)|$ in (x_{i-1}, x_i) by G_i, then from our definition, we have

$$\sum_{i=1}^{n} (x_i - x_{i-1}) G_i < \varepsilon,$$

hence

$$\int_a^b |f(x) - \varphi(x)|\,dx < \varepsilon.$$

For the sake of brevity, we shall call such a function $\varphi(x)$ as defined in the lemma an *associate* function of $f(x)$ with respect to ε in the interval (a, b).

II.

Theorem II. The necessary and sufficient conditions that

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy$$

exists, whenever $u(x)$ *is bounded in the interval* (a, ∞), *are that:*

i. $$\lim_{a\to\infty} a(x, y) \text{ exists,} \quad y \geqq a,$$

ii. $$\lim_{x\to\infty}\int_a^x |\, a(x, y)\,|\, dy \text{ exists, and } =\int_a^\infty |\, b(y)\,|\, dy,$$

where

$$b(y)=\lim_{x\to\infty} a(x, y).$$

When the conditions are fulfilled, we have

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy=\int_a^\infty b(y)\, u(y)\, dy.$$

Before proving this theorem we shall insert the following

Lemma 4. If the function $a(x, y)$ *satisfies conditions* i *and* ii *in Theorem II, then*

$$\lim_{x\to\infty}\int_a^x |\, c(x, y)\,|\, dy=0,$$

where

$$c(x, y)=a(x, y)-b(y).$$

As, by Lemma 2 (*a*), $\int_a^\infty |\, b(y)\,|\, dy$ converges, we have for sufficiently large quantity X

$$\int_x^\infty |\, b(y)\,|\, dy < \frac{\varepsilon}{8}, \quad x > X,$$

where ε is a given arbitrary small positive quantity, and by condition ii,

$$\int_a^x |\, a(x, y)\,|\, dy < \int_a^\infty |\, b(y)\,|\, dy+\frac{\varepsilon}{8}, \quad x > X_1$$

$$< \int_a^X |\, b(y) \,|\, dy + \int_X^\infty |\, b(y) \,|\, dy + \frac{\varepsilon}{8},$$

$$< \int_a^X |\, b(y) \,|\, dy + \frac{\varepsilon}{4}, \quad x > \text{Max. } (X, X_1).$$

On the other hand,

$$\int_a^x |\, a(x, y) \,|\, dy = \int_a^X |\, a(x, y) \,|\, dy + \int_X^x |\, a(x, y) \,|\, dy.$$

Since by Lemma 1 $\lim\limits_{x \to \infty} a(x, y) = b(y)$ uniformly in y, $a \leqq y \leqq X$, we have

$$\int_a^X |\, a(x, y) \,|\, dy > \int_a^X |\, b(y) \,|\, dy - \frac{\varepsilon}{4}, \quad x > X_2.$$

Therefore

$$\int_a^X |\, b(y) \,|\, dy + \frac{\varepsilon}{4} > \int_a^X |\, a(x, y) \,|\, dy > \int_a^X |\, b(y) \,|\, dy$$

$$+ \int_X^x |\, a(x, y) \,|\, dy - \frac{\varepsilon}{4}, \quad x > \text{Max. } (X, X_1, X_2),$$

thence

$$\int_X^x |\, a(x, y) \,|\, dy < \frac{\varepsilon}{2}, \quad x > \text{Max. } (X, X_1, X_2);$$

accordingly

$$\int_X^x |\, c(x, y) \,|\, dy \leqq \int_X^x |\, a(x, y) \,|\, dy + \int_X^x |\, b(y) \,|\, dy$$

$$< \frac{3\,\varepsilon}{4}.$$

And, as $\lim\limits_{x \to \infty} c(x, y) = 0$ uniformly in y, $a \leqq y \leqq X$,

$$\int_a^X |\, c(x, y) \,|\, dy < \frac{\varepsilon}{4}, \quad x > X_3;$$

hence, we have

$$\int_a^x |\, c(x, y) \,|\, dy < \varepsilon, \quad x > \text{Max. } (X_1, X_2, X_3),$$

so that

$$\lim_{x \to \infty} \int_a^x |\, c(x, y) \,|\, dy = 0.$$

Now we proceed to the proof of Theorem II.

Proof for sufficiency.

If we denote the upper limit of $|u(x)|$ in (a, ∞) by U, then

$$\left|\int_a^x c(x, y)\, u(y)\, dy\right| < U\int_a^x |c(x, y)|\, dy.$$

Since by Lemma 4,

$$\lim_{x\to\infty}\int_a^x |c(x, y)|\, dy=0,$$

$$\lim_{x\to\infty}\int_a^x c(x, y)\, u(y)\, dy=0,$$

and by Lemma 1,

$$\int_a^\infty b(y)\, u(y)\, dy$$

converges, we have

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy=\lim_{x\to\infty}\left\{\int_a^x b(y)\, u(y)\, dy+\int_a^x c(x, y)\, u(y)\, dy\right\}$$

$$=\int_a^\infty b(y)\, u(y)\, dy.$$

Proof for necessity.

As $u(x)$ here considered contains all the functions considered in Theorem I, the conditions

$$\lim_{x\to\infty} a(x, y)=b(y), \quad y\geqq a,$$

$$\int_a^x |a(x, y)|\, dy < M$$

are necessary.

Therefore, by Lemma 2 (a) $\int_a^\infty |b(y)|\, dy$ converges.

Write

$$\overline{\lim_{x\to\infty}}\int_a^x |a(x, y)|\, dy=A$$

and

$$\int_a^\infty |b(y)|\, dy=B,$$

then, assuming that

$$A \neq B,$$

we have by Lemma 2 (b)

$$A > B.$$

Denote $A-B$ by 5γ, then $\gamma > 0$, and

$$\overline{\lim_{x\to\infty}} \int_b^x |a(x,y)|\, dy = \overline{\lim_{x\to\infty}} \left\{ \int_a^x |a(x,y)|\, dy - \int_a^b |a(x,y)|\, dy \right\}$$
$$> A - B$$
$$> 5\gamma.$$

Fix a number x_1 such that the inequality

$$\int_b^{x_1} |a(x_1, y)|\, dy > 3\gamma$$

holds. Denote an associate function of $a(x_1, y)$ with respect to γ in the interval (b, x_1) by $\bar{a}(x_1, y)$, and determine $u(x)$ such that

$$|u(x)| = 1, \quad \operatorname{sgn} u(x) = \operatorname{sgn} \bar{a}(x_1, y), \quad b < x \leqq x_1 .$$

Then

$$\int_b^{x_1} |a(x_1, y) - \bar{a}(x_1, y)|\, dy < \gamma,$$

so that

$$\int_b^{x_1} |\bar{a}(x_1, y)|\, dy > 2\gamma,$$

and

$$\left| \int_b^{x_1} \{a(x_1, y) - \bar{a}(x_1, y)\}\, u(y)\, dy \right| \leqq \int_b^{x_1} |a(x_1, y) - \bar{a}(x_1, y)|\, dy < \gamma,$$

namely

$$\left| \int_b^{x_1} a(x_1, y)\, u(y)\, dy - \int_b^{x_1} |\bar{a}(x_1, y)|\, dy \right| < \gamma.$$

Therefore

$$\int_b^{x_1} a(x_1, y)\, u(y)\, dy > \gamma.$$

By Lemma 1, we have

$$\lim_{x\to\infty} \int_b^{x_1} |a(x, y)|\, dy = \int_b^{x_1} |b(y)|\, dy < \gamma,$$

so that

$$\int_b^{x_1} |a(x,y)|\,dy < \gamma, \quad x > X_1.$$

Fix $x_2 (> X_1)$ such that

$$\int_{x_1}^{x_2} |a(x_2,y)|\,dy > 4\gamma$$

holds. Denote an associate function of $a(x_2, y)$ with respect to γ in the interval (x_1, x_2) by $\bar{a}(x_2, y)$, and determine $u(x)$ such that

$$|u(x)| = 1, \quad \operatorname{sgn} u(x) = -\operatorname{sgn} \bar{a}(x_2, y), \quad x_1 < x \leqq x_2.$$

Then we have

$$\int_{x_1}^{x_2} |\bar{a}(x_2,y)|\,dy > 3\gamma$$

and

$$\left| \int_{x_1}^{x_2} \{a(x_2,y) - \bar{a}(x_2,y)\}\, u(y)\,dy \right| \leqq \int_{x_1}^{x_2} |a(x_2,y) - \bar{a}(x_2,y)|\,dy < \gamma,$$

i. e.

$$\left| \int_{x_1}^{x_2} a(x_2,y)\, u(y)\,dy + \int_{x_1}^{x_2} |\bar{a}(x_2,y)|\,dy \right| < \gamma,$$

hence

$$\int_{x_1}^{x_2} a(x_2,y)\, u(y)\,dy < -2\gamma,$$

so that

$$\begin{aligned}\int_b^{x_2} a(x_2,y)\, u(y)\,dy &< \int_b^{x_1} a(x_2,y)\, u(y)\,dy - \int_{x_1}^{x_2} a(x_2,y)\, u(y)\,dy \\ &< \gamma - 2\gamma \\ &< -\gamma.\end{aligned}$$

Further, fix X_2 and $x_3 (> X_2)$ such that

$$\int_b^{x_2} |a(x,y)|\,dy < \gamma, \quad x > X_2$$

and

$$\int_{x_2}^{x_3} |a(x_3,y)|\,dy > 4\gamma$$

hold respectively, and denote an associate function of $a(x_3, y)$ with respect to γ in the interval (x_2, x_3) by $\bar{a}(x_3, y)$, and determine $u(x)$ such that

$$|u(x)|=1, \quad \operatorname{sgn} u(x)=\operatorname{sgn} \bar{a}(x_3, y), \quad x_2<x \leqq x_3,$$

then we have similarly as above

$$\int_b^{x_3} a(x_3, y)\, u(y)\, dy>\gamma.$$

Proceeding in this way we can define a sequence of numbers $\{x_n\}$ for which $\lim\limits_{x\to\infty} x_n=\infty$, and a bounded function $u(x)$, such that

$$\int_b^{x_{2m+1}} a(x_{2m+1}, y)\, u(y)\, dy>\gamma$$

and

$$\int_b^{x_{2m}} a(x_{2m}, y)\, u(y)\, dy<-\gamma$$

hold for every integer m.

On the other hand, as the number of the discontinuous points of the function $u(x)$ is finite, $u(x)$ is integrable in any finite interval, and we have

$$\overline{\lim_{x\to\infty}}\int_a^x a(x, y)\, u(y)\, dy>\int_a^b b(y) u(y) dy+\gamma,$$

$$\underline{\lim_{x\to\infty}}\int_a^x a(x, y)\, u(y)\, dy<\int_a^b b(y) u(y) dy-\gamma,$$

so that

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy$$

does not exist.

Therefore it must be

$$\int_a^\infty |b(y)|\, dy=\overline{\lim_{x\to\infty}}\int_a^x |a(x, y)|\, dy.$$

But, since by Lemma 2 (b)

$$\int_a^\infty |b(y)|\, dy\leqq\underline{\lim_{x\to\infty}}\int_a^x |a(x, y)|\, dy,$$

$\lim\limits_{x\to\infty}\int_a^x |a(x, y)|\, dy$ must exist and

$$\lim_{x\to\infty}\int_a^x |a(x, y)|\, dy=\int_a^\infty |b(y)|\, dy.$$

III.

Theorem III. The necessary and sufficient conditions that

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy$$

exists, whenever $u(x)$ *is monotonic and* $\lim\limits_{x\to\infty} u(x)$ *exists, are that:*

i. $$\lim_{x\to\infty} a(x, y) \ \textit{exists}, \quad y \geqq a,$$

ii. $$\lim_{x\to\infty}\int_a^x a(x, y)\, dy \quad \textit{exists},$$

iii. $$\left| \int_a^z a(x, y)\, dy \right| < M, \quad a \leqq z \leqq x,$$

where M *is a positive quantity independent of* x *and* z.

When the conditions are fulfilled, we have

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy = \lim_{x\to\infty} u(x) \int_a^x a(x, y)\, dy + \int_a^\infty b(y)\{u(y) - \lim_{x\to\infty} u(x)\}\, dy,$$

where

$$b(y) = \lim_{x\to\infty} a(x, y).$$

Proof for sufficiency.

Similarly as in the proof of Lemma 2 (a), we can show that $\left| \int_a^x b(y)\, dy \right|$ is limited.

Write

$$\lim_{x\to\infty} u(x) = l$$

and

$$u(x) - l = w(x),$$

then $w(x)$ is monotonic, and

(7) $$\lim_{x\to\infty} w(x) = 0.$$

Hence, by a theorem on integral (1),

$$\int_a^\infty b(y)\, w(y)\, dy$$

---

(1) See Bromwich, Theory of infinite series, p. 430.

converges.

Now

$$\int_a^x a(x, y)\, u(y)\, dy = l\int_a^x a(x, y)\, dy + \int_a^x a(x, y)\, w(y)\, dy$$
$$= l\int_a^x a(x, y)\, dy + \int_a^x b(y)\, w(y)\, dy + \int_a^x c(x, y)\, w(y)\, dy,$$

where

$$c(x, y) = a(x, y) - b(y);$$

and

$$\left| \int_a^z c(x, y)\, dy \right| \leqq \left| \int_a^x a(x, y)\, dy \right| + \left| \int_a^z b(y)\, dy \right|$$
$$< M'.$$

Since by

$$(7) \qquad | w(x) | < \frac{\varepsilon}{2M'}, \quad x > X,$$

it follows

$$\left| \int_X^x c(x, y)\, w(y)\, dy \right| < \frac{\varepsilon}{2M'}\, 2M' \text{ [1]}$$
$$< \varepsilon, \quad x > X,$$

also by Lemma 1, we have

$$\left| \int_a^X c(x, y)\, w(y)\, dy \right| < \varepsilon, \quad x > X'.$$

Therefore

$$\lim_{x \longrightarrow \infty} \int_a^x c(x, y)\, w(y)\, dy = 0,$$

so that

$$\lim_{x \longrightarrow \infty} \int_a^x a(x, y)\, u(y)\, dy$$

exists, and we have

$$\lim_{x \longrightarrow \infty} \int_a^x a(x, y)\, u(y)\, dy = l \lim_{x \longrightarrow \infty} \int_a^x a(x, y)\, dy + \int_a^\infty b(y)\, w(y)\, dy$$
$$= \lim_{x \longrightarrow \infty} u(x) \int_a^x a(x, y)\, dy + \int_a^\infty b(y)\, w(y)\, dy.$$

(1) See Bromwich, loc. cit., p. 426.

Proof for necessity.

(a) Let $u(x)$ be a monotonic function, and put

$$\lim_{x\to\infty} u(x)=l, \quad u(x)-l=w(x);$$

then $w(x)$ is also a monotonic function converging to zero.

Since

$$\lim_{x\to\infty}\int_a^x a(x, y)\, u(y)\, dy \quad \text{and} \quad \lim_{x\to\infty}\int_a^x a(x, y)\, w(y)\, dy$$

exist,

$$\lim_{x\to\infty}\int_a^x a(x, y)\, dy$$

must exist.

Thus the condition i is necessary.

(b) Let $u(x)$ be a monotonic decreasing function, and define a function $\bar{u}(x)$, such that

$$\begin{aligned} \bar{u}(x)&=1+u(x), & a\leqq x\leqq y,\\ \bar{u}(x)&=\phantom{1+{}}u(x), & x>y, \end{aligned}$$

then $\bar{u}(x)$ is also a monotonic decreasing function.

Write

$$f(x, y)=\int_a^x a(x, s)\, \bar{u}(s)\, ds-\int_a^x a(x, s)\, u(s)\, ds=\int_a^y a(x, s)\, ds,$$

then

$$\lim_{x\to\infty} f(x, y)$$

must exist.

Denote

$$\frac{f(x, y+r)-f(x, y)}{r} \quad \text{by} \quad \varphi(x, r),$$

then

$$\lim_{r\to 0}\varphi(x, r)=\frac{\partial f(x, y)}{\partial y}=a(x, y).$$

From the assumption regarding to the uniform continuity of $a(x, y)$, it is easily shown that $\varphi(x, r)$ approaches uniformly to $a(x, y)$ for all $x\geqq h$, when r tends to zero, hence the limit of $\varphi(x, r)$ for $x\to\infty$, $r\to 0$ exists, so that

$$\lim_{x\to\infty} a(x, y)=\lim_{x\to\infty}\lim_{r\to 0}\varphi(x, r)$$

exists. Thus the condition ii is necessary.

(c) Write

$$\left| \int_a^z a(x,y)\, dy \right| = S(x,z), \quad a \leqq z \leqq x.$$

Assume that $S(x,z)$ be not limited, and denote the upper limit of $S(x,z)$ with respect to z for each x by $G(x)$ and the first point in which $S(x,z)$ takes (1) the value of $G(x)$ by Z_x, then by assumption

$$\overline{\lim_{x\to\infty}}\, G(x) = \infty,$$

and by (b) and Lemma 1,

$$\overline{\lim_{x\to\infty}}\, Z_x = \infty.$$

Let a be a number greater than 3, and write

$$\int_a^z |b(y)|\, dy = \beta(z).$$

Fix x_1 such that

$$G(x_1) > a^3;$$

then by (b) and Lemma 1, we have

$$\int_a^{z_{x_1}} |a(x,y)|\, dy < \beta(z_{x_1}) + a, \quad x > X_1.$$

Next fix x_2 which is greater than X_1 and satisfies the inequality

$$G(x_2) > a^5 + a^3 + a^2 + (a^2 + a)\,\beta(z_{x_1}).$$

Similarly fix X_2 and $x_3 (> X_2)$ for which

$$\int_a^{z_{x_2}} |a(x,y)|\, dy < \beta(z_{x_2}) + a, \quad x > X_2$$

and

$$G(x_3) > a^7 + a^4 + a^2 + (a^3 + a)\,\beta(z_{x_2})$$

hold respectively, and so on.

Thus we have the following two sequences of numbers

$$x_1,\ x_2,\ x_3,\ \cdots\cdots$$

and

$$X_1,\ X_2,\ X_3,\ \cdots\cdots$$

---

(1) As the function $S(x,z)$ is continuous in z for each x.

satisfying the relations

$$\lim_{i\to\infty} x_i = \infty ,$$

$$x_i > X_{i-1} , \quad i = 1, 2, 3, \cdots\cdots ,$$

$$\int_a^{zx_{i-1}} |\, a(x, y)\,|\, dy < \beta(zx_{i-1}) + a, \quad x > X_{i-1}$$

and

$$G(x_i) > a^{2i+1} + a^{i+1} + a^2 + (a^i + a)\,\beta(zx_{i-1}).$$

Define a function $u(x)$, such that

$$u(x) = \frac{1}{a}, \quad a \leqq x < zx_1 ,$$

$$u(x) = \frac{1}{a^i}, \quad zx_{i-1} \leqq x < zx_i , \quad i = 2, 3, \cdots\cdots .$$

Since

$$\int_a^{x_i} a(x_i, y)\,u(y)\,dy = \int_a^{zx_{i-1}} a(x_i, y)\,u(y)\,dy + \frac{1}{a^i}\int_{zx_{i-1}}^{zx_i} a(x_i, y)\,dy$$

$$+ \int_{zx_i}^{x_i} a(x_i, y)\,u(y)\,dy,$$

$$\left| \int_a^{x_i} a(x_i, y)\,u(y)\,dy \right| \geqq \frac{1}{a^i}\left\{ G(x_i) - \int_a^{zx_{i-1}} |\, a(x_i, y)\,|\, dy \right\}$$

$$- \frac{1}{a}\int_a^{zx_{i-1}} |\, a(x_i, y)\,|\, dy - \frac{2\,G(x_i)}{a^{i+1}}$$

$$\geqq \frac{(a-2)\,G(x_i)}{a^{i+1}} - \left(\frac{1}{a^i} + \frac{1}{a}\right)\{\beta(zx_{i-1}) + a\}$$

$$> \frac{1}{a^{i+1}}\{a^{2i+1} + a^{i+1} + a^2 + (a^i + a)\,\beta(zx_{i-1})\}$$

$$- \left(\frac{1}{a^i} + \frac{1}{a}\right)\{\beta(zx_{i-1}) + a\}$$

$$> a^i ,$$

so that

$$\lim_{x\to\infty} \left| \int_a^{x_i} a(x_i, y)\,u(y)\,dy \right| = \infty .$$

On the other hand, from our definition $u(x)$ is integrable and decreasing [1], and

$$\lim_{x \to \infty} u(x) = 0.$$

Thus the assumption regarding to $S(x, z)$ leads us to a contradiction, hence the condition iii is necessary.

IV.

In Theorem I or III, *if we confine the function $u(x)$ only to what converges to zero, when x tends to infinity, then the conditions* i *and* iii *are necessary and sufficient.* Also we have

Theorem IV. If

$$\lim_{x \to \infty} \int_a^x a(x, y)\, u(y)\, dy$$

exists, whenever $\lim_{x \to \infty} u(x) = \alpha$, *where α is a fixed number different from zero, then*

$$\lim_{x \to \infty} \int_a^x a(x, y)\, u(y)\, dy$$

always exists whenever $\lim_{x \to \infty} u(x)$ *exists.*

Quite a similar theorem holds good for the case where $u(x)$ is a monotonic function.

Feb. 10th, 1918.

---

(1) By a slight modification, we can easily define a steadily decreasing function for the same purpose.

On the Mean Centre of Points on an Algebraic Curve,

by

Kitizi Yanagihara, Sendai.

The object of this note is, in the first place, to arrange a brief list of papers relating to the theorems of Chasles and their allied subjects, and in the second, to add to them some analogues as the applications of Liouville's theorem on elimination by which he proved the theorem of Chasles.

I

The so-called theorem of Chasles is:

If the complete system of tangents (tangent planes) parallel to a given straight line (plane) be drawn to an algebraic curve (surface), the mean centre of the points of contact is independent of the direction of the given line (plane)(1).

Chasles proved this first by a modified Newton's theorem on the diameter of curves(2), and afterwards analytically(3).

In 1841, Liouville elegantly proved, in his "Mémoire sur quelques propositions générales de géomètrie et sur la théorie de l'élimination dans les équations algébriques"(4), the following important theorem (referred to this theorem by the symbol L_1 later on.)

Theorem. If we put two polynomials in x, y

$$M(x, y)=0, \qquad N(x, y)=0$$

in the form

$$\begin{aligned} M(x, y) \equiv a_{10}\, x^m + a_{11}\, x^{m-1}\, y + \cdots\cdots + a_{1m}\, y^m \\ + a_{21}\, x^{m-1} + a_{22}\, x^{m-2}\, y + \cdots\cdots + a_{2m}\, y^{m-1} \end{aligned}$$

---

(1) Sur la transformation parabolique des relations métriques des figures. Quetelet, Correspondence mathematique et physique, tome VI., 1830, p. 10.

(2) Salmon: Higher Plane curves, 3rd ed., § 137.

(3) Chasles: Géométrie supérieure, 1st ed., pp. 358-360.

(4) Liouville's Journal, 1st series, vol. VI, p. 359. See also Serret's Cours d'Algèbre, 6th ed., vol. I, chapter V.

$$+\cdots\cdots$$
$$\equiv x^m f(u)+x^{m-1}f_1(u)+x^{m-2}f_2(u)+\cdots\cdots=0, \qquad (1)$$
$$N(x, y)\equiv b_{10}x^n+b_{11}x^{n-1}y+\cdots\cdots+b_{1n}y^n$$
$$+b_{21}x^{n-1}+b_{22}x^{n-2}y+\cdots\cdots+b_{2n}y^{n-1}$$
$$+\cdots\cdots$$
$$\equiv x^n F(u)+x^{n-1}F_1(u)+x^{n-2}F_2(u)+\cdots\cdots=0, \qquad (2)$$

where $ux=y$, then the sum of the roots of the resultant to which the elimination of y from (1) and (2) leads is expressed by the formula

$$\Sigma x=-\Sigma\frac{\beta F'(a)+F_1(a)}{F(a)}, \qquad (3)$$

where $$f(a)=0, \qquad \beta f'(a)+f_1(a)=0,$$

the summation on the right hand side extending over all the roots of the eqution $f(a)=0$.

He has also proved an analogous result for the case of three polynomials in three variables, and then as their application, proved the theorems due to Chasles.

Of Liouville's result (3), it should be observed that Σx depends exclusively upon the first two terms of both (1) and (2). From this fact he deduced the following theorem (L_2):

The mean centre of the points of intersection of $M(x, y)=0$ and $N(x, y)=0$ remains fixed, if $N(x, y)=0$ vary keeping all its asmpytotes unchanged.

In this memoir will also be found some results due to Duhamel, of which we will cite one below.

If we draw a complete system of parallel tangents to an algebraic curve, the sum of the radii of curvature at the points of contact is always zero, whatever may be the direction of the tangents, the radii of curvature being distinguished as positive or negative according as the point of contact is downwardly convex or upwardly convex toward the tangent.

In 1887, G. Humbert published a paper "Sur le théorème d'Abel et quelques-unes de ses applications géométriques" in which he proved a theorem, as the fundamental one, for the sum of the variations of an arbitrarily given Abelian integral appertaining to a given algebraic curve $f(x, y)=0$, along the arcs described by the points of intersection

of $f=0$ with a variable algebraic curve belonging to a pencil(1).

In the same year, succeeding to the previous paper, he published another one "Sur quelques propriétés des courbes" in which he expounded an elementary method, and deduced various results(2).

Among them is found mentioned the following theorem as a particular case of Liouville's theorem (L_2).

The mean centre of points of intersection of $f=0$ with a circle having a fixed point as centre is independent of the magnitude of its radius, from which he then, by remembering the fact that the mean centre of an algebraic curve $f=0$ with another algebraic curve $g=0$ is unchanged, if we replace $g=0$ by all its asymptotes, deduced the following theorem.

The mean centre of points of intersection of $f=0$ with a circle is that of the projections of the centre of the circle upon the asymptotes of $f=0$.

In 1887, another paper "Sur un théorème de Chasles" written by Weill also made its appearance a little before the last paper of Humbert, in the same volume of the journal(3). In this, Weill proved in a refreshing manner the theorem of Chasles for the plane curve.

In 1890, H. M. Taylor published a paper "On the centre of an algebraic curve"(4) in which he proved, by defining the centre of a a curve as the mean centre of the points of contact of parallel tangents, the following results.

All curves which have the same foci have the same centre.

The centre of a curve is the mean centre of the foci.

All parallel curves have the same foci.

In the same year 1890, Fouret gave a proof, in his "Démonstratoin et applications d'un théorème de Liouville sur l'élimination(5), to the following theorem which can be looked upon as an extension of Liouville's result (L_1).

In the resultant

$$A_0 x_1^t + A_1 x_1^{t-1} + A_2 x_1^{t-2} + \cdots\cdots = 0,$$

(1) Liouville's Journal, 4th Series, Vol. 3.
(2) Nouv. Ann. de Math., 3rd series, Vol. VI, p. 526.
(3) P. 82.
(4) Quarterly Journal of Math., Vol. 24, p. 55.
(5) Nouv. Ann. de Math. 3rd series,, vol. IX, p. 258.

$$t=m_1 . m_2 . m_3 \cdots\cdots m_\lambda,$$

to which the elimination of $\lambda-1$ variables from

$$u_{p0}+u_{p1}+u_{p2}+\cdots\cdots=0,$$
$$p=1, 2, 3, \cdots\cdots, \lambda,$$

leads, the coefficients A_k exclusively depend upon those in

$$u_{p0},\ u_{p1},\ u_{p2}, \cdots\cdots u_{pk}, \qquad p=1, 2, 3, \cdots\cdots, \lambda,$$

u_{ps} being a homogenous polynomial of degree m_p-s in $x_1, x_2, \cdots\cdots, x_\lambda$; and among many other interesting results, will be found the following:

If an algebraic curve having no parabolic branch vary, keeping its asymptotes fixed, the mean centre of the points, the normals at which are concurrent, is a fixed point.

In 1906, Laurent, in his "Sur une théorème de Chasles et d'Abel"(1), extended the theorem of Chasles on surface to that in n-dimensional space, and showed that this theorem is a special case of Abel's theorem.

In 1915, by applying Liouville's results (3), A. Pleskot tried to find a class of algebraic curves having the property that the mean centre of the points of contact of the tangents drawn from an arbitrary point remains fixed, and showed that if the equation of the curve $\phi(x, y)=0$ can be put in the form

$$V_n(x, y)+V_{n-3}(x, y)+V_{n-4}(x, y)+\cdots\cdots=0, \qquad (5)$$

V_k being a homogeneous polynomial of degree k in x, y, and if the roots of

$$f(a)\equiv V_n(1, a)=0$$

be all distinct, then $\phi(x, y)=0$ is a required one(2).

II.

As already mentioned, Liouville's result (3) shows us at once that Σx depends only upon the first two terms in (1) and (2). Hence we get, as immediate consequences, the following theorems.

1. *Let C be the mean centre of points of an algebraic curve $f(x, y)=0$, at which the length of the tangent is t, then the position of C is independent of t.*

(1) Nouv. Ann. de Math., 4th series, vol. VI, p. 266.

(2) Monatshefte für Math. u. Phys., Vol. 26, p. 135.

For, since we have

$$t=y\sqrt{1+y'^2}/y'$$

we get

$$y'^2=y^2/(t^2-y^2).$$

Eliminating y' from this and

$$f_x+y'f_y=0,$$

we get

$$y^2(f_x^2+f_y^2)-t^2f_x^2=0,$$

in which the degrees of $y^2(f_x^2+f_y^2)$ and f_x^2 differ by two.

2. *Next let C be the mean centre of points of $f(x, y)=0$, at which the normals have the length n, then the position of C is independent of n.*

For, since we have

$$n=y\sqrt{1+y'^2},$$

we get

$$y'^2=(n^2-y^2)/y^2.$$

Eliminating y' from this and

$$f_x+y'f_y=0,$$

we get

$$y^2(f_x^2+f_y^2)-n^2f_y^2=0.$$

By a similar way, we can prove analogous theorems for the sub-tangent, the sub-normal, the segment on any fixed straight line intercepted by the tangent and normal at the same point, and the area enclosed by any fixed straight line, and the tangent and normal at the same point.

3. *Let C be the mean centre of points, real or imaginary, on an algebraic curve $f(x, y)=0$, having ρ as the radius of curvature, then the position of C is independent of the radius ρ.*

For, since we have

$$\rho=-\frac{(f_x^2+f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2},$$

we get

$$(f_x^2+f_y^2)^3-\rho^2(f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2)=0.$$

4. If the osculating circle to a curve cut a circle

$$X^2+Y^2=r^2,$$

at an angle θ, we have

$$2\rho r\cos\theta=r^2+\rho^2-d^2,$$

d being the distance between the centres of these two circles. Then we have

$$2r\frac{(1+y'^2)^{\frac{2}{3}}}{y''}\cos\theta=r^2+\frac{(1+y'^2)^3}{y''^2}-\left[\left(x-\frac{y'(1+y'^2)}{y''}\right)^2+\left(y+\frac{1+y'^2}{y''}\right)^2\right],$$

whence we get

$$A+2r^2B+r^4C^4=0,$$

where

$$\begin{aligned}
A&\equiv[(f_x^2+f_y^2)^3-\{f_x(f_x^2+f_x^2)-xC\}^2-\{f_y(f_x^2+f_y^2)-yC\}^2]^2,\\
B&\equiv C^2[(f_x^2+f_y^2)^3-\{f_x(f_x^2+f_y^2)-xC\}^2-\{f_y(f_x^2+f_y^2)-yC\}^2],\\
&\quad-2C^2\cos^2\theta(f_x^2+f_y^2)^3,\\
C&\equiv f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2.
\end{aligned}$$

Hence we get the following

Theorem. Let C be the mean centre of points of $f(x,y)=0$, the osculating circles at which cut a circle with radius r at an angle θ, whose centre is a fixed point, then the position of C is independent of both r and θ.

5. If the centre of the osculating circle be denoted by I, the condition that the osculating circle should touch the three sides of any triangle inscribed in a circle with radius r whose centre is the origin of coordinates is

$$OI^2=r^2\pm2r\rho,$$

that is,

$$\left(y+\frac{1+y'^2}{y''}\right)^2+\left(x-\frac{y'(1+y'^2)}{y''}\right)^2=r^2\pm2r\frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

Whence we get

$$M-2r^2N+r^4C^4=0,$$

where

$$M\equiv[\{f_y(f_x^2+f_y^2)-yC\}^2+\{f_xf_y(f_y-f_x)-xC\}^2]^2,$$

$$N\equiv 2C^2(f_x^2+f_y^2)^3+C^2[\{f_y(f_x^2+f_y^2)-yC\}^2+\{f_xf_y(f_y-f_x)-xC\}^2],$$

$$C\equiv f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2.$$

Hence we get the following

Theorem. Let C be the mean centre of points of $f(x,y)=0$, the osculating circles at which touch the three sides of any triangle inscribed in a circle with radius r whoes centre is a fixed point, then the position of C is independent of r.

Similarly we will have the following

Theorem. Let C be the mean centre of points of $f(x,y)=0$, the osculating circle which is circumscribed about any triangle circumscribing about a circle with radius r, whose centre is a fixed point, then the position of C is independent of r.

6. Hitherto we have constantly made use of Liouville's theorem on elimination, in what follows, however, Fouret's extension will play the fundamental rôle.

Let $\phi(X,Y)=0$ be an algebraic curve belonging to a system of curves having all its asymptotes, real or imaginary, in common, and if the normal to $\phi=0$ at (X,Y) touch $f=0$ at (x,y), then we get

$$f(x,y)=0,$$

$$\phi(X,Y)=0, \tag{1}$$

$$(x-X)\frac{\partial\phi}{\partial Y}-(y-Y)\frac{\partial\phi}{\partial X}=0, \tag{2}$$

$$(x-X)\frac{\partial f}{\partial x}+(y-Y)\frac{\partial f}{\partial y}=0.$$

Thus we have four relations connecting four numbers x, y, X, Y, and the variable element in (1) and (2) does not appear in the two highest homogeneous parts. Hence by Fouret's extension of Liouville's theorem, the mean centre of points (x,y), as well as that of points (X,Y), is invariable in whatever manner may $\phi=0$ vary.

This theorem, however, fails if the highest homogeneous part of $\phi=0$ is symmetrical with respect to X, Y.

7. If a central conic $\phi=0$ osculate an algebraic curve $f(x,y)=0$ at (x,y), and the centre of $\phi=0$ lie on another algebraic curve $U=0$, we get

$$f(x,y)=0,\quad \frac{df}{dx}=0,\quad \frac{d^2f}{dx^2}=0,\quad \frac{d^3f}{dx^3}=0,\quad \frac{d^4f}{dx^4}=0,$$

$$\phi(x,y)=0,\quad \frac{d\phi}{dx}=0,\quad \frac{d^2\phi}{dx^2}=0,\quad \frac{d^3\phi}{dx^3}=0,\quad \frac{d^4\phi}{dx^4}=0,$$

$$\frac{\partial\phi(\alpha,\beta)}{\partial\alpha}=0\qquad \frac{\partial\phi(\alpha,\beta)}{\partial\beta}=0,$$

$$U(\alpha,\beta)=0.$$

We have thus 13 relations connecting 13 numbers x, y, y', y'', y''', $y^{(4)}, \alpha, \beta$ and five arbitrary constants in $\phi=0$. Hence, if $U(\alpha,\beta)=0$ vary keeping all its asymptotes fixed, by Fouret's extension of Liouville's theorem, the mean centre of the centres of the osculating conic $\phi=0$, as well as that of the point of $f=0$, at which $\phi=0$ osculates $f=0$, remains fixed.

8. *Let C be the mean centre of points of $f=0$, at which the latus rectum of the osculating parabola $\phi=0$ is p, then the position of C is independent of p.*

For, if $\phi=0$ osculates $f=0$ at (x,y), we get

$$f(x,y)=0,\quad \frac{df}{dx}=0,\quad \frac{d^2f}{dx^2}=0,\quad \frac{d^3f}{dx^3}=0,$$

$$\phi(x,y)=0,\quad \frac{d\phi}{dx}=0,\quad \frac{d^2\phi}{dx^2}=0,\quad \frac{d^3\phi}{dx^3}=0.$$

The radius of curvature ρ of a point (α,β) of $\phi(\alpha,\beta)=0$ is given by

$$\beta''^2\rho^2=(1+\beta'^2)^3. \tag{1}$$

Hence, if (α,β) be its vertex, we must have

$$3\beta'\beta''^2=(1+\beta'^2)\beta''',$$

together with

$$\phi(\alpha,\beta)=0,\quad \frac{d\phi}{d\alpha}=0,\quad \frac{d^2\phi}{d\alpha^2}=0,\quad \frac{d^3\phi}{d\alpha^3}=0.$$

We have thus 14 relations connecting 14 numbers x, y, y', y'', y''', $\alpha, \beta, \beta', \beta'', \beta'''$ and four arbitrary constants in $\phi=0$, and ρ^2 or $p^2/4$ appears only in (1), and the degree of $(1+\beta'^2)^3$ exceeds that of β''^2 by 4.

Hence the theorem is proved.

The foregoing proof shows at the same time that the mean centre

of points at which the parabola osculates $f=0$ is also independent of the magnitude of the latus rectum p.

9. We will next try to extend the theorem in 1 which can be stated as follows.

If on the tangent at P of $f(x, y)=0$, we take two points M and N so that

$$PM=PN=t,$$

and the locus of M and N be denoted by (f, t), then the mean centre of the common points of (f, t) and any given straight line l is independent of t.

We will now show that the straight line l can be replaced by an algebraic curve $\phi(X, Y)=0$.

Let the tangent at P of $f(x, y)=0$ cut $\phi=0$ in $Q, Q', Q'', \cdots\cdots$, and translate the vectors $PQ, PQ', PQ'', \cdots\cdots$ into $OT, OT', OT'', \cdots\cdots$, O being the origine of coordinates.

If now the coordinates of Q, T be denoted by (X, Y) and (α, β) respectively, then we have

$$\phi(X, Y)=0, \qquad f(x, y)=0,$$

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0,$$

$$\beta : \alpha=-\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y},$$

$$\alpha^2+\beta^2=(X-x)^2+(Y-y)^2.$$

If we eliminate x, y, X, Y from these five equations, we get $F(\alpha, \beta)=0$ which is algebraic in α, β as the equation to the locus of $Q, Q', Q''\cdots\cdots$.

Now, for the points of intersection of $\phi(X, Y)=0$ with (f, t), we have

$$X=x+t\cos\theta, \qquad Y=y+t\,\mathrm{nis}\,\theta,$$

so that

$$\Sigma X=\Sigma x+t\Sigma\cos\theta, \qquad \Sigma Y=\Sigma y+t\Sigma\sin\theta,$$

θ being the angle made by the positive direction of x-axis and the vector PQ.

Professors Kakeya and Kubota have noticed that ΣX and ΣY are constant, since (f, t) has all its asymptotes in common with $f=0$.

Then it is sufficient to show that $t\Sigma\cos\theta$ and $t\Sigma\sin\theta$ are independent of t.

To this end, we consider the intersection of $F(\alpha, \beta)=0$ and the circle $\alpha^2+\beta^2=t^2$. The sums of the coordinates are evidently

$$t\Sigma \cos\theta, \qquad t\Sigma \sin\theta.$$

But, as Humbert mentioned in his paper referred to in § 1, these two quantities are independent of t. Hence Σx and Σy are also independent of t.

This method of proof has been suggested by a proof, by which Prof. Hayashi showed that in the theorem in § 3, the mean centre of the centres of curvature is also independent of ρ (consult his paper in this Journal, vol. 14, p. 98).

On the Mean Centre of the Contact Points of Tangent Planes to an Algebraic Surface,

by

KWAN SHIBATA, Sendai.

Mr. A. Pleskot has got a certain sufficient condition under which the mean centre of the contact points of tangents to a plane algebraic curve drawn through a given point on its plane is independent of the position of that point(1). His investigation seems to have been suggested by the well-known theorem due to Chasles(2): If tangents having the same direction be drawn to a plane algebraic curve, the mean centre of their contact points is independent of the direction.

In what follows, I will try to solve the two analogous problems in space: When tangent planes forming an axial pencil are drawn to an algebraic surface, what conditions are sufficient in order that the mean centre of their contact points does not depend on

(i) the position of the axis of the pencil, but depending on its direction only, or

(ii) the direction of the axis of the pencil, but depending on its position only?

1. I will begin with case (i).

Let an algebraic surface of degree n, referred to a rectangular Cartesian coordinate system, be

$$S(x, y, z)=0.$$

Then the contact points of the tangent planes forming an axial pencil whose axis passes through the point (ξ, η, ζ) and has the direction (p, q, r) are the intersections of the following three surfaces

$$S=0,$$

$$p\frac{\partial S}{\partial x}+q\frac{\partial S}{\partial y}+r\frac{\partial S}{\partial z}=0,$$

---

(1) A. Pleskot, Über das Zentrum der mittleren Abstände der Berührungspunkte an algebraischen Kurven, Monatshefte für Math. u. Physik, Bd. 26 (1915), S. 135.

(2) Chasles, Aperçu historique des méthodes en géométrie, p. 624.

$$(\xi-x)\frac{\partial S}{\partial x}+(\eta-y)\frac{\partial S}{\partial y}+(\zeta-z)\frac{\partial S}{\partial z}=0,$$

which can be written in the forms

$$(1)\qquad \begin{cases} \sum\limits_{i=0}^{n} f_i(x,y,z)=0, \\ \sum\limits_{i=0}^{n-1} \varphi_i(x,y,z)=0, \\ \sum\limits_{i=0}^{n-1} \psi_i(x,y,z)=0, \end{cases}$$

where f_i, φ_i, ψ_i are homogeneous expressions of x, y, z of degrees $n-i$, $n-i-1$, $n-i-1$ respectively, and which can be again transformed into

$$(2)\qquad \begin{cases} \sum\limits_{i=0}^{n} x^{n-i} f_i(u,v)=0\text{(}^1\text{)}, \\ \sum\limits_{i=0}^{n-1} x^{n-i-1}\varphi_i(u,v)=0, \\ \sum\limits_{i=0}^{n-1} x^{n-i-1}\psi_i(u,v)=0, \end{cases}$$

where

$$(3)\begin{cases} \varphi_i=(n-i)pf_i(u,v)-(pu-q)\dfrac{\partial f_i}{\partial u}-(pv-r)\dfrac{\partial f_i}{\partial v}, \\ \psi_i=(n-i)\xi f_i(u,v)-(\xi u-\eta)\dfrac{\partial f_i}{\partial u}-(\xi v-\zeta)\dfrac{\partial f_i}{\partial v}+(i+1)f_{i+1}, \end{cases}$$

if we put u, v for y/x, z/x respectively. Eliminating y, z from (1) we have an equation in x, of degree $n(n-1)^2$, whose roots are the x-coordinates of the contact points. Hence the x-coordinate of their mean centre is the sum of the roots of this equation divided by $n(n-1)^2$, so that we need only to consider the coefficient, with the sign changed, of the second term divided by that of the first term, supposing that the terms of the equation are arranged in the order of descending powers of x. Thus, if we denote the quotient by Σx,

$$(4)\qquad \Sigma x=-\Sigma\left\{\frac{\partial\psi}{\partial\alpha}\alpha'+\frac{\partial\psi}{\partial\beta}\beta'+\psi_1(\alpha,\beta)\right\}\Big/\psi(\alpha,\beta)\text{(}^2\text{)},$$

---

(1) Throughout this paper, I use for brevity the symbol $\chi(u, v)$ for the quotient of a homogeneous expression $\chi(x, y, z)$ divided by the highest power of x in which then y/x, z/x are replaced by u, v respectively.

(2) J. Liouville, Mémoire sur quelques propositions générales de géométrie et sur la théorie de l'élimination dans les équations algébriques. Liouville Jour. t. 6 (1841), 345.

the summation being extended over the solutions (α, β) of the set of equations

$$f(u, v)=0, \qquad \varphi(u, v)=0,$$

i.e.

$$(5) \qquad \begin{cases} f(u, v)=0, \\ (pu-q)\dfrac{\partial f}{\partial u}+(pv-r)\dfrac{\partial f}{\partial v}=0, \end{cases}$$

while (α', β') are the solutions of the set of equations

$$\frac{\partial f}{\partial \alpha}\alpha'+\frac{\partial f}{\partial \beta}\beta'+f_1=0,$$

$$\frac{\partial \varphi}{\partial \alpha}\alpha'+\frac{\partial \varphi}{\partial \beta}\beta'+\varphi_1=0,$$

i.e.

$$(6) \quad \begin{cases} \dfrac{\partial f}{\partial \alpha}\alpha'+\dfrac{\partial f}{\partial \beta}\beta'+f_1=0, \\ \dfrac{\partial f}{\partial \beta}\left\{\dfrac{\partial^2 f}{\partial \alpha^2}\alpha'+\dfrac{\partial^2 f}{\partial \alpha \partial \beta}\beta'+\dfrac{\partial f_1}{\partial \alpha}\right\}-\dfrac{\partial f}{\partial \alpha}\left\{\dfrac{\partial^2 f}{\partial \alpha \partial \beta}\alpha'+\dfrac{\partial^2 f}{\partial \beta^2}\beta'+\dfrac{\partial f_1}{\partial \beta}\right\}=0\text{(1)}. \end{cases}$$

Evidently α and β as well as α' and β' depend on p, q, r and not on ξ, η, ζ.

On account of (3), (5) and (6), equality (4) becomes

$$(7) \qquad \Sigma x=-\Sigma\frac{AP+BR}{A\dfrac{\partial f}{\partial \alpha}+Bf_1}=-\Sigma\frac{AQ-CR}{A\dfrac{\partial f}{\partial \beta}-Cf_1},$$

where

$$A=(\xi\beta-\zeta)(p\alpha-q)-(\xi\alpha-\eta)(p\beta-r),$$

$$B=p\beta-r,$$

$$C=p\alpha-q,$$

$$P=\frac{\partial^2 f}{\partial \alpha^2}\alpha'+\frac{\partial^2 f}{\partial \alpha \partial \beta}\beta'+\frac{\partial f_1}{\partial \alpha},$$

$$Q=\frac{\partial^2 f}{\partial \alpha \partial \beta}\alpha'+\frac{\partial^2 f}{\partial \beta^2}\beta'+\frac{\partial f_1}{\partial \beta},$$

---

(1) Here and hereafter, such a symbol as $\dfrac{\partial^{i+k}\chi}{\partial \alpha^i\,\partial \beta^k}$ stands for the value of $\dfrac{\partial^{i+k}\chi(u, v)}{\partial u^i\,\partial v^k}$ $(i, k=0, 1, 2)$ for $u=\alpha,\ v=\beta$.

$$R = \frac{\partial f_1}{\partial \alpha}\alpha' + \frac{\partial f_1}{\partial \beta}\beta' + 2f_2.$$

Hence, if Σx is independent of the position of $P(\xi\,\eta\,\zeta)$, then since it must have the same value as that at the origin of coordinates

$$\Sigma x = -\Sigma R/f_1 .$$

Hence if, by equating the corresponding terms on both sides, we assume that

$$\frac{AP+BR}{A\dfrac{\partial f}{\partial \alpha}+Bf_1} = \frac{R}{f_1}, \qquad \text{i.e.} \quad \frac{P}{\dfrac{\partial f}{\partial \alpha}} = \frac{R}{f_1},$$

and accordingly by (7)

$$\frac{P}{\dfrac{\partial f}{\partial \alpha}} = \frac{Q}{\dfrac{\partial f}{\partial \beta}} = \frac{R}{f_1},$$

then Σx has always the same value independent of (ξ, η, ζ). This condition may be written

$$(8) \qquad \begin{vmatrix} \begin{vmatrix} \dfrac{\partial^2 f}{\partial \alpha^2} & \dfrac{\partial f}{\partial \alpha} \\ \dfrac{\partial f_1}{\partial \alpha} & f_1 \end{vmatrix} . & \begin{vmatrix} \dfrac{\partial^2 f}{\partial \alpha \partial \beta} & \dfrac{\partial f}{\partial \beta} \\ \dfrac{\partial f_1}{\partial \beta} & f_1 \end{vmatrix} & \begin{vmatrix} \dfrac{\partial f_1}{\partial \alpha} & 2f_2 \\ \dfrac{\partial f}{\partial \alpha} & f_1 \end{vmatrix} \\ \dfrac{\partial f}{\partial \alpha} & \dfrac{\partial f}{\partial \beta} & f_1 \\ \dfrac{\partial\left(f, \dfrac{\partial f}{\partial \alpha}\right)}{\partial(\alpha, \beta)} & \dfrac{\partial\left(f, \dfrac{\partial f}{\partial \beta}\right)}{\partial(\alpha, \beta)} & \dfrac{\partial(f, f_1)}{\partial(\alpha, \beta)} \end{vmatrix} = 0.$$

Next transform (1) into

$$\sum_{i=0}^{n} y^{n-i} \ f_i\ (u_1, v_1) = 0,$$

$$\sum_{i=0}^{n-1} y^{n-i-1}\, \varphi_i\ (u_1, v_1) = 0,$$

$$\sum_{i=0}^{n-1} y^{n-i-1}\, \psi_i\ (u_1, v_1) = 0,$$

where $$x/y=u_1, \quad z/y=v_1;$$

and proceed as before, then we have

$$\Sigma y=-\Sigma\left\{\frac{\partial\psi}{\partial\alpha_1}\alpha_1'+\frac{\partial\psi}{\partial\beta_1}\beta_1+\psi_1\right\}/\psi(\alpha_1,\beta_1),$$

where (α_1, β_1) is a solution of the set of equations

$$f(u_1, v_1)=0, \qquad \varphi(u_1, v_1)=0,$$

and (α_1', β_1') is that of the set

$$\frac{\partial f}{\partial\alpha_1}\alpha_1'+\frac{\partial f}{\partial\beta_1}\beta_1'+f_1=0,$$

$$\frac{\partial\varphi}{\partial\alpha_1}\alpha_1'+\frac{\partial\varphi}{\partial\beta_1}\beta_1'+\varphi_1=0,$$

and arrive at

$$\frac{P_1}{\dfrac{\partial f}{\partial\alpha_1}}=\frac{Q_1}{\dfrac{\partial f}{\partial\beta_1}}=\frac{R_1}{f_1}$$

as a sufficient condition for independency of Σy from (ξ, η, ζ), P_1, Q_1, R_1 being the expressions got by replacing α_1, β_1 for α, β in P, Q, R respectively. But it can be easily shown that this condition is equivalent to that for Σx, by paying attention to relations

$$f_i(u_1, v_1)=u^{-(n-i)}f_i(u, v),$$

$$u_1=u^{-1} \quad \text{and} \quad v_1=u^{-1}v.$$

A similar reasoning is applicable to Σz.

So we have

Theorem I. If the equation to an algebraic surface satisfy condition (8) for any solution of the set of equations (5), the mean centre of the contact points of tangent planes to the surface, passing through a straight line having given direction cosines, is independent of the position of the straight line: or in other words, the mean centres of the contact points are the same for all axial pencils of tangent planes to the surface, if the axes of the pencils be parallel to one another.

2. Now let us proceed to case (ii).

As before, we have

$$\Sigma x = -\Sigma\left\{\frac{\partial\varphi}{\partial\lambda}\lambda' + \frac{\partial\varphi}{\partial\mu}\mu' + \varphi_1\right\} / \varphi(\lambda, \mu),$$

$$= -\Sigma\frac{DS+ET}{D\dfrac{\partial f}{\partial\lambda}+E\dfrac{\partial f}{\partial\mu}} \quad \text{or} \quad -\Sigma\frac{AT-DU}{A\dfrac{\partial f}{\partial\mu}-Df_1},$$

where

$$D = p\lambda - q,$$

$$E = p\mu - r,$$

$$S = \frac{\partial^2 f}{\partial\lambda^2}\lambda' + \frac{\partial^2 f}{\partial\lambda\partial\mu}\mu' + \frac{\partial f_1}{\partial\lambda},$$

$$T = \frac{\partial^2 f}{\partial\lambda\partial\mu}\lambda' + \frac{\partial^2 f}{\partial\mu^2}\mu' + \frac{\partial f_1}{\partial\mu},$$

$$U = \frac{\partial f_1}{\partial\lambda}\lambda' + \frac{\partial f_1}{\partial\mu}\mu' + 2f_2,$$

(λ, μ) being a solution of the set of equations

$$f(u, v) = 0, \qquad \psi(u, v) = 0,$$

i.e.

$$(9') \qquad \begin{cases} f(u, v) = 0, \\ (\xi u - \eta)\dfrac{\partial f}{\partial u} + (\xi v - \zeta)\dfrac{\partial f}{\partial v} - f_1 = 0, \end{cases}$$

and (λ', μ') that of the set of equations

$$\frac{\partial f}{\partial\lambda}\lambda' + \frac{\partial f}{\partial\mu}\mu' + f_1 = 0,$$

$$\frac{\partial\psi}{\partial\lambda}\lambda' + \frac{\partial\psi}{\partial\mu}\mu' + \psi_1 = 0,$$

i.e.

$$\frac{\partial f}{\partial\lambda}\lambda' + \frac{\partial f}{\partial\mu}\mu' + f_1 = 0,$$

$$(\xi\lambda - \eta)S + (\xi\mu - \zeta)T - U = 0.$$

If Σx is independent of the direction (p, q, r), then Σx must have the same value as that for $(p=0,\ q=0)$ or $(p=0,\ r=0)$. So

$$(10)\qquad \Sigma x=-\Sigma\frac{S}{\dfrac{\partial f}{\partial\lambda}}=-\Sigma\frac{T}{\dfrac{\partial f}{\partial\mu}}=-\Sigma\frac{U}{f_1}.$$

Hence we have, for a sufficient condition for the independency of Σx from (p, q, r),

$$\frac{S}{\dfrac{\partial f}{\partial\lambda}}=\frac{T}{\dfrac{\partial f}{\partial\mu}}=\frac{U}{f_1};$$

whence, in the form free from λ', μ', we have

$$(11)\qquad \begin{vmatrix} \begin{vmatrix} \dfrac{\partial^2 f}{\partial\lambda^2} & \dfrac{\partial f}{\partial\lambda} \\ \dfrac{\partial f_1}{\partial\lambda} & f_1 \end{vmatrix} & \begin{vmatrix} \dfrac{\partial^2 f}{\partial\lambda\partial\mu} & \dfrac{\partial f}{\partial\lambda} \\ \dfrac{\partial f_1}{\partial\mu} & f_1 \end{vmatrix} & \begin{vmatrix} \dfrac{\partial f_1}{\partial\lambda} & 2f_2 \\ \dfrac{\partial f}{\partial\lambda} & f_1 \end{vmatrix} \\ \dfrac{\partial f}{\partial\lambda} & \dfrac{\partial f}{\partial\mu} & f_1 \\ \dfrac{\partial\left(f, \dfrac{\partial f}{\partial\lambda}\right)}{\partial(\lambda, \mu)} & \dfrac{\partial\left(f, \dfrac{\partial f}{\partial\mu}\right)}{\partial(\lambda, \mu)} & \dfrac{\partial(f, f_1)}{\partial(\lambda, \mu)} \end{vmatrix}=0.$$

It can be easily shown that, under this condition Σy, Σz are also independent of (p, q, r).

So we have

Theorem 2. If the equation to an algebraic surface satisfy condition (11) for any solution of the set of equations (9), the mean centre of the contact points of tangent planes to the surface through a straight line which passes through a fixed point, is independent of the direction of the straight line. In other words: If an axial pencil of tangent planes, with a ray of a line-bundle having a given vertex as the axis of the pencil, be drawn to an algebraic surface satisfying condition (11), the mean centre of the contact points is definite for the bundle.

Remark. Take another point $P'(\xi', \eta', \zeta')$ on the axis of the pencil considered, and let (ξ, η, ζ), (ξ', η', ζ') go to infinity at the same time, the ratios $\xi:\eta:\zeta$ and $\xi':\eta':\zeta'$ kept as constants, then the axis of the pencil goes to infinity and all the tangent planes become parallel, p, q, r becoming indeterminate. In this special case, we see by (10)

$$\Sigma x = -\Sigma \frac{S}{\dfrac{\partial f}{\partial \lambda}} \quad \text{or} \quad -\Sigma \frac{T}{\dfrac{\partial f}{\partial \mu}} \quad \text{or} \quad -\Sigma \frac{U}{f_1}.$$

This is independent of the direction of the tangent planes, and the following theorem also due to Chasles is proved: If a system of tangent planes parallel to a given plane be drawn to an algebraic surface, the mean centre of their contact points is independent of the direction of the given plane.

July, 1917.

二ツノ代數曲線ノ交點ノ平均中心ニ就テ

On the Mean Centre of the Points of Intersection of Two Algebraic Curves,

林　鶴　一（仙臺）

Tsuruichi Hayashi, Sendai.

1. 柳原君ノ調査ニヨルニ G. Humbert 教授ハ Liouville ノ包括的定理ノ系トシテ次ノ定理ヲ證明シタリ ([1]). 一系ヲナセル同心圓ト一ツノ任意ノ代數曲線トヲ交ハラシメタルトキ，各圓上ノ交點ニ同一ノ質量アリト見テ夫レ等ノ交點ノ重心ヲ求ムレバ，其ノ位置ハ其圓ノ半徑ニ關セズ一定ナリ. 今此ノ面白キ定理ノ別證明ヲ揭ゲントス.

直角坐標ノ原點ヲ同心圓ノ中心ニ置キ n 次ノ代數曲線ノ方程式ヲ

$$f(x,y)=u_n+u_{n-1}+u_{n-2}+\cdots\cdots=0 \qquad (1)$$

トス，但シ u_r ハ x ト y トノ r 次ノ同次整式ナリトス.

半徑ガ ρ ナル圓ノ周上ノ點ノ坐標ハ

$$x=\rho\cos\theta=\rho\frac{e^{2i\theta}+1}{2\,e^{i\theta}},$$

$$y=\rho\sin\theta=\rho\frac{e^{2i\theta}-1}{2\,i\,e^{i\theta}}$$

ナリ. 故ニ

$$\rho\,e^{i\theta}=z$$

ト置カバ

$$x=\frac{z^2+\rho}{2\,z}, \qquad y=\frac{z^2-\rho}{2\,i\,z}.$$

之ヲ方程式 (1) 中ニ代入シテ z^n ヲ乘ズレバ

$$v_n+z\,v_{n-1}+z^2\,v_{n-2}+\cdots\cdots=0$$

([1]) Nouvelles Annales de Math. (3) 6, 1887. p. 535. (本誌第 14 卷第 80 頁ニ於ケル柳原君ノ論文參照).

但シ v_r ハ u_r 中ノ x, y ニ代ユルニ夫々 $(z^2+\rho)/2$, $(z^2-\rho)/2i$ ヲ以テシタルモノナリ. 故ニ v_r ヲスベテ z^2 ノ遞降冪ノ順序ニ排列スレバ

$$(A_0 z^{2n}+A_1 z^{2n-2}+\cdots\cdots)+(B_0 z^{2n-1}+B_1 z^{2n-3}+\cdots\cdots)+\cdots\cdots=0.$$

而シテ係數 $A_1, \cdots\cdots, B_1, \cdots\cdots$ ハ ρ ヲ含ムモ A_0 ト B_0 トハ ρ ヲ含マズ.

故ニ此 $2n$ 次ノ方程式ヲ滿足スル z ノ値即チ彼ノ代數曲線ト圓トノ交點ノ坐標 x, y ノ値ハ

$$\Sigma z=\Sigma x+i\,\Sigma y=-B_0/A_0$$

ヲ滿足シテ ρ ニ關係セズ. 故ニ Σx 及ビ Σy モ亦 ρ ニ關係セズ.

極メテ特段ナル場合トシテ一ツノ楕圓

$$b^2 x^2+a^2 y^2=a^2 b^2$$

ト, 中心 (α, β) ガ一定シテ半徑 r ガ媒介變數ナル同心圓ノ一系

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

ニ屬スル一圓トノ交點ノ重心ヲ求ムルニ, 其坐標ハ

$$\frac{a^2\alpha}{a^2-b^2}, \qquad \frac{b^2\beta}{b^2-a^2}$$

ナルコトヲ知ル, 即チ r ト獨立ナリ. 偖テ又此ノ坐標ノ値ガ a ト b トノ絕對値ニ關係ナク, 其比, 從ツテ楕圓ノ離心率ニノミ關係スルコトニ注意スベシ, 後節更ニ此事ニ及バン.

又此ノ重心ノ坐標ノ式ニヨリテ, 同心圓ノ中心ガ一ツノ曲線ヲ畫クトキハ重心ノ畫ク曲線ハ容易ニ求メラル. タトヘバ前者ガ圓ナルトキハ後者ハ所設ノ楕圓ト共軸ナル楕圓ナリ.

2. 一ツノ代數曲線ノ切線ノ正ノ向キト同ジ向キヲ有シ, 其切點ニ於ケル曲率半徑ト同ジ長サヲ有スル動徑ヲ引ケバ其端ハ亦一ツノ代數曲線ヲ畫ク.

何トナレバ所設ノ代數曲線ノ切線ノ正ノ向キヲ x 軸ノ正ノ向キヨリ計リテ θ ニテ表ハシ, 其切點ニ於ケル曲率半徑ヲ ρ ニテ表ハセバ, 上ノ動徑ノ端ノ坐標ハ

$$\left.\begin{aligned} x_1&=\rho\cos\theta=\rho\frac{dx}{\sqrt{dx^2+dy^2}}=\frac{(f_x'^2+f_y'^2)f_y'}{f_{xx}''f_y'^2-2f_{xy}''f_x'f_y'+f_{yy}''f_x'^2} \\ y_1&=\rho\sin\theta=\rho\frac{dx}{\sqrt{dx^2+dy^2}}=\frac{-(f'^2_x+f'^2_y)f_x'}{f_{xx}''f_y'^2-2f_{xy}''f_x'f_y'+f_{yy}''f_x'^2} \end{aligned}\right\}\quad(2)$$

但シ所設ノ代數曲線ノ方程式ヲ

$$f(x, y)=0 \tag{1}$$

トス．然ラバ (2) ノ右邊ハ共ニ x, y ノ有理函數ナリ．故ニ (1) ト (2) ノ兩式トノ間ニ, x, y ヲ消去スレバ, 彼ノ動徑ノ端ノ軌跡トシテ

$$f_1(x_1, y_1)=0$$

ノ如キ方程式ヲ得, 但シ f_1 ハ x_1 ト y_1 トノ有理整函數トス．此ノ軌跡ヲ ρ ガ一定ノ値ヲ有スル圓ニテ截レバ其ノ交點ノ重心ノ位置ガ ρ ノ値ニ關係セザルコトハ前ニ證明セリ．故ニ

$$\Sigma \rho \cos\theta, \quad 及 \quad \Sigma \rho \sin\theta$$

ハ ρ トハ獨立ナリ．然ルニ所設ノ代數曲線ノ曲率中心ノ坐標ヲ求ムレバ

$$\xi=x-\rho \sin\theta, \qquad \eta=y+\rho\cos\theta.$$

然ルニ柳原君ノ證明スル所ニヨリ ρ ガ一定スレバ (1)

$$\Sigma x \quad 及 \quad \Sigma y$$

ハ ρ トハ獨立ナリ．故ニ

$$\Sigma \xi \quad 及 \quad \Sigma \eta$$

モ亦然リ．故ニ次ノ**定理**ヲ得．

一ツノ代數曲線ニ於テ曲率半徑ガ一定ノ値ヲ有スル諸點ニ應ズル曲率中心ノ重心ノ位置ハ其ノ曲率半徑ノ値ニ關係セズ．

此ノ證明ノ途中ニ於テ得タル,

$$\Sigma \rho \cos\theta \quad 及 \quad \Sigma \rho \sin\theta$$

ノ ρ トハ獨立ナルコトヲ**定理**ノ形ニ於テ述ブレバ次ノ如シ．

一ツノ代數曲線ニ於テ曲率半徑ガ一定ノ値ヲ有スル諸點ヲ取リ, 此等ノ諸點ヲ各其曲率中心ト連結セル曲率半徑ノ, 任意ノ方向ニ投ゼル正射影ノ平均値ハ, 其曲率半徑ノ値ニ關係セズ．

3. 第 1 節ト同樣ノ方法ニヨリテ一ツノ代數曲線ト一系ヲナセル相似ニシテ相似ノ位置ニアル同心ノ楕圓又ハ雙曲線トノ交點ノ重心ヲ研究スルヲ得．

---

(1) 曲線上ノ一點ニ於ケル切線ノ長サ又ハ法線ノ長サヲ, 其ノ點ヨリ, 其ノ切線又ハ法線ガ一定直線ト出會ヘル點ニ至ルマデノ距離トスレバ．柳原君ノ此ノ定理ノ證明ト同樣ノ趣向ニヨリテ次ノ**定理**ヲ證明シ得ベシ．

一ツノ代數曲線上ニアリテ, 切線ノ長サ又ハ法線ノ長サガ, 絕對値ニ於テ相等シキ諸點ノ重心ハ其等ノ長サノ絕對値ニ係ハラズ一定不易ナリ．

今 λ ヲ媒介變數トシテ斯ノ如キ楕圓ノ一系ノ方程式ヲ

$$\frac{x^2}{a^2}+\frac{y^2}{\lambda^2 a^2}=1$$

トスレバ

$$x=a\cos\theta=a\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{z^2+a^2}{2z},$$

$$y=\lambda a\sin\theta=\lambda a\frac{e^{i\theta}-e^{-i\theta}}{2i}=\lambda\frac{z^2-a^2}{2iz},$$

但シ $z=ae^{i\theta}$ トス.

故ニ前ノ同心圓ノ系ノ如クシテ所設ノ代數曲線トノ交點ニ於テハ

$$\Sigma z$$

ガ a トハ獨立ナリ. 其實數部ト虛數部トヲ離セバ

$$\Sigma a\cos\theta, \quad \Sigma a\sin\theta,$$

從ツテ

$$\Sigma x, \quad \Sigma y$$

ハ a トハ獨立ナリ.

λ ノ代ハリニ $i\lambda$ トセハ雙曲線ノ一系ヲ取扱ヒシコトヽナル.

又此節ノ事項ハ正射影ニヨリテ同心圓ガ相似ニシテ相似ノ位置ニアル同心ノ有心二次曲線トナリ, 代數曲線ハ猶ホ代數曲線トナルコトニ注目スレバ第 1 節ノ事項ニヨリテ自カラ明瞭ナリ.

從ツテ第 1 節ノ末項ニヨリテ次ノ面白キ**定理**ヲ得

相似ニシテ相似ノ位置ニアル同心ノ有心二次曲線ノ二系アルトキ, 一系ニ屬スル任意ノ一ツト他系ニ屬スル任意ノ一ツトノ交點ノ重心ハ, 各系ヨリ何レヲ選出ストモ一定不易ナリ. 即チ其ノ二系ニ固有ノ定點ナリ.

又相似ニシテ相似ノ位置ニアル同心ノ直角雙曲線系ト, 同心圓ノ一系トアルトキ, 各系ヲ一ツヅヽ選出シテ得ル四交點ノ重心ハ, 兩中心ヲ連ヌル線分ノ中點ナリ.

又本節ノ楕圓系ニ於テ $\lambda=\infty$ ト置ケバ, 其系ハ y 軸ニ平行ニシテ之ヨリ等距離ニアル二直線ノ系トナル. 故ニ次ノ**定理**アリ.

一ツノ代數曲線ト,一定直線ヨリ等距離ニアル二直線トノ交點ノ重心ハ,其ノ距離ノ如何ニ拘ハラズ一定不易ナリ.

コハ掛谷君ガ他ノ方面ヨリ得ラレタル結果ヨリモ導カルヽモノナルガ,殆ド證明ナシニ明白ナル定理ナリ.

4. 或種ノ圓錐曲線ト同心圓ノ一系トノ交點ノ重心ニ就キテハ次ノ如ク證明スルモ可ナリ. 圓錐曲線ノ直角でかると坐標ガ媒介變數 t ニヨリテ

$$x=a_0 t^2+a_1 t+a_2,$$

$$y=b_0 t^2+b_1 t+b_2$$

トスルコトヲ得ルトキ同心圓ノ一系

$$(x-\alpha)^2+(y-\beta)^2=r^2$$

ヲ取レバ, 其ノ交點ニ對スル媒介變數 t ノ値 t_1, t_2, t_3, t_4 ハ四次方程式

$$(a_0 t^2+a_1 t+a_2-\alpha)^2+(b_0 t^2+b_1 t+b_2-\beta)^2=r^2$$

ヲ滿足ス. 故ニ

$$\Sigma t_1=-(2a_0 a_1+b_0 b_1)\div(a_0^2+b_0^2),$$

$$\Sigma t_1 t_2=\{a_1^2+2 a_0 (a_2-\alpha)+b_1^2+2 b_0 (b_2-\beta)\}\div(a_0^2+b_0^2).$$

故ニ Σt_1 及ビ Σt_1^2 ハ r ニ關係セズ.

然ルニ

$$\Sigma x_1=a_0 \Sigma t_1^2+a_1 \Sigma t_1+4 a_2,$$

$$\Sigma y_1=b_0 \Sigma t_1^2+b_1 \Sigma t_1+4 b_2.$$

故ニ Σx_1 及ビ Σy_1 モ亦 r ニ關係セズ.

同樣ナル論法ニヨリテ方程式

$$x=a_0 t^m+a_1 t^{m-1}+\cdots\cdots+a_{m-1} t+a_m,$$

$$y=b_0 t^n+b_1 t^{n-1}+\cdots\cdots+b_{n-1} t+b_n$$

ヲ有スル代數曲線ト同心圓ノ一系トノ交點ニ就テ論ズルコトヲ得. $m=1$, $n\geqq 2$ ナル場合ニ於テハ圓ノ中心ノ位置及半徑ノ如何ニ拘ハラズ $\Sigma t_1=0$ トナリテ $\Sigma x_1=0$ ナリ. 故ニ交點ノ重心ハ y 軸ニ平行ナル定直線ノ上ニアリ. コハ近頃 F. Irwin 及 H. N. Wright ノ兩氏ガ Annals of Mathematics, Vol. 19, 1917, pp. 152–158 ニ於テ揭ゲタル所謂 Polynomial curves

$$y=p_0x^n+p_2x^{n-2}+\cdots\cdots+p_n$$

ト任意ノ直線トノ交點ノ重心ニ關スル定理ト多少ノ關聯ヲ有ス.

又斯クシテ方程式

$$a^{m-n}y^n=x^m,$$

$$y^nx^m=a^{n+m}$$

ヲ有スル高次拋物線又ハ高次雙曲線ニ關スル定理ヲ得ベシ.

更ニ空間曲線

$$x=a_0t^l+a_1t^{l-1}+\cdots\cdots+a_{l-1}t+a_l,$$

$$y=b_0t^m+b_1t^{m-1}+\cdots\cdots+b_{m-1}t+b_m,$$

$$z=c_0t^n+c_1t^{n-1}+\cdots\cdots+c_{n-1}t+c_n$$

ト同心球ノ一系トノ交點ニツキテモ同様ニ論ズルコトヲ得. 就中或種ノ空間三次曲線ハ $l=m=n=3$ ノ場合トシテ此ノ形ノ方程式ニテ表ハシ得ルハ明白ノ事實ナルガ故ニ此種ノ空間三次曲線ト同心球ノ一系トノ交點ノ重心ハ其球ノ半徑ニ關セズ一定ノ位置ヲ有スルコトヲ知ル.

大正六年十月

Über die Konstruktionsaufgaben dritten und vierten Grades,

von

TADAHIKO KUBOTA in Sendai.

1. Es ist vom analytischen Standpunkte eine wohlbekannte Tatsache (1), dass alle Konstruktionsaufgaben dritten oder vierten Grades nur mit Lineal und Zirkel sich auf die nachstehende Fundamentalaufgabe zurückführen lassen: *Wenn zwei Kegelschnitte durch die auf ihnen gelegenen Punktquintupeln gegeben sind, gesucht sei die vier Schnittpunkte der beiden Kegelschnitte zu bestimmen.* Es wurde schon bewiesen, dass *diese Fundamentalaufgabe nur mit Lineal und Zirkel lösbar ist, wenn ein Kegelschnitt, der kein Kreis ist, oder eine rationale kubische Kurve oder eine rationale biquadratische Kurve vorgezeichnet gegeben ist* (2). Doch scheint es mir wünschenswert, eine systematische Methode zu haben, wie man einzelne Aufgaben dritten und vierten Grades auf die Fundamentalaufgabe in geometrischer Weise reduzieren kann. Nun lassen sich alle Aufgaben dritten und vierten Grades nach geringer Modifikation auf die Aufgaben der Bestimmung der Koinzidenzpunkte der $(1, 2)$, $(2, 2)$, oder $(1, 3)$ deutigen Korrespondenz der aufeinander liegenden Punktreihen reduzieren, wie die Aufgaben zweiten Grades sich auf die Aufgaben der Bestimmung der Koinzidenzpunkte der aufeinander liegenden projectiven Punktreihen zurückführen lassen.

Wir beschränken uns, unbeschadet der Allgemeinheit, auf die Betrachtung der aufeinander liegenden Punktreihen auf einem Kreis.

Die Bestimmung der Koinzidenzpunkte der $(1, 2)$ deutigen Korrespondenz der aufeinander liegenden Punktreihen des Kreises lässt sich in bekannter Weise auf die Fundamentalaufgabe wie folgt reduzieren: Es seien B_1, B_2 die beiden Punkte, welche dem Punkte A entsprechen. Beschreibt der Punkt A die Punktreihe auf dem Kreise, so beschreibt

(1) Dies wurde schon von Descartes in seiner „Géométrie“ angegeben.

(2) H. Kortum, Mémoire sur quelques problèms cubiques et biquadratiques, Annali di mat., 1869. F. London, Die geometrischen Konstruktionen dritten und vierten Grades, Zeitschrift für Math. u. Phys., 1896. T. Kubota, Über die Erweiterung des Smith-Kortumschen Satzes u. s. w., dieses Journ., 5, 1914.

das entsprechende Punktepaar B_1, B_2 eine involutorische Punktreihe, sodass die Verbindungsgerade B_1B_2 durch einen und denselben Punkt P hindurchgeht. Dann beschreiben die Geraden OA, PB_1B_2 zwei projektive Strahlenbüschel. Also erzeugt der Schnittpunkt der Geraden OA, PB_1B_2 einen Kegelschnitt, welcher durch die Punkte O, P hindurchgeht. Die drei anderen Schnittpunkte des Kegelschnitts mit dem Kreise sind also die gesuchten Punkte. Damit ist unsere Aufgabe auf die Fundamentalaufgabe zurückgeführt.

Nun gehen wir zur Bestimmung der Koinzidenzpunkte der $(2, 2)$ deutigen Korrespondenz auf einem Kreise über.

Projiziert man die entsprechenden Punkte A, B bzw. aus den Punkten O, O' des Kreises, so erzeugt der Schnittpunkt der beiden entsprechenden Strahlen OA, $O'B$ eine Kurve vierter Ordnung mit den Doppelpunkten O, O'. Diese Kurve hat vier weitere Punkte mit dem Kreise gemein, welche nichts anderes als die gesuchten Koinzidenzpunkte sind. Damit ist unsere Aufgabe auf die Folgende zurückgeführt. *Wenn eine Kurve vierter Ordnung mit den reellen Doppelpunkten O, O' durch acht weitere Punkte gegeben und ein Kegelschnitt durch die fünf Punkte O, O', A_1, A_2, A_3 gegeben sind, gesucht sei vier andere Schnittpunkte der beiden Kurven zu bestimmen.*

Nimmt man einen Punkt P auf der Kurve vierter Ordnung und wendet die Steinersche Transformation in Bezug auf das Dreieck POO' auf die Figur an, so werden die Kurve vierter Ordnung und der Kegelschnitt bzw. in eine kubische Kurve und einen Kegelschnitt transformiert, welche durch die Punkte O, O' hindurchgehen. Damit ist unsere Aufgabe auf die Folgende reduziert: *Wenn eine kubische Kurve durch neun Punkte $O, O', A, B, 1, 2, 3, 4, 5$ und ein Kegelschnitt durch fünf Punkte O, O', P, Q, R gegeben sind, gesucht sei vier weitere Schnittpunkte der beiden Kurven zu bestimmen.*

Um diese Aufgabe zu lösen benutzen wir die Chasles-Schrötersche Erzeugungsweise der kubischen Kurve (¹). Man denke nun die fünf Kegelschnitte $(O, O', A, B, 1)$, $(O, O', A, B, 2)$, $(O, O', A, B, 3)$, $(O, O', A, B, 4)$, $(O, O' A, B, 5)$ und bestimme dann den Punkt X, sodass die Projektivität

$$((O, O', A, B), 1, 2, 3, 4, 5) \barwedge X(1, 2, 3, 4, 5)$$

entsteht, wenn man sich die fünf Kegelschnitte bzw. auf die fünf Strahlen beziehen lässt. Dadurch kann man sich irgend einen Kegelschnitt des

(¹) Schröter, Die Theorie der ebenen Kurven dritter Ordnung, 1888. Chasles, Comptes Rendus 34.

Kegelschnittbüschels auf einen Strahl des Strahlenbüschels beziehen lassen. Dann beschreiben die beiden Schnittpunkte des Kegelschnitts sowie des entsprechenden Strahls die in Betracht gezogene kubische Kurve. Nun kann man nur mit Lineal und Zirkel die Schnittpunkte M_1, N_1; M_2, N_2; M_3, N_3; M_4, N_4; M_5, N_5 der Kegelschnitte $(O, O', A, B, 1)$, $(O, O', A, B, 2)$, $(O, O', A, B, 3)$, $(O, O', A, B, 4)$, $(O, O', A, B, 5)$ mit dem gegebenen Kegelschnitte bestimmen. Dann gehen die Verbindungsgeraden $M_1 N_1$, $M_2 N_2$, $M_3 N_3$, $M_4 N_4$, $M_5 N_5$ nach dem Steinerschen Satze durch einen und denselben Punkt hindurch, sodass $PM_1 N_1$, $PM_2 N_2$, $PM_3 N_3$, $PM_4 N_4$, $PM_5 N_5$ zu den Strahlen $X1$, $X2$, $X3$, $X4$, $X5$ projektivisch zugeordnet werden können. Diese beiden projektivisch aufeinander bezogenen Strahlenbüschel erzeugen einen Kegelschnitt. Die vier Schnittpunkte dieses Kegelschnitts mit dem gegebenen Kegelschnitte sind also die gesuchten Schnittpunkte. Damit ist unsere Aufgabe auf die Fundamentalaufgabe zurückgeführt.

2. In dem Falle, wo das involutorische Punktepaar durch unsere $(2, 2)$ deutige Korrespondenz dem involutorischen Punktepaare entspricht, kann man unsere Aufgabe in bekannter Weise noch einfacher auflösen. Es seien A_1, A_2; B_1, B_2 entsprechende Punktepaare der beiden Involutionen auf einem Kreise. Dann gehen die Geraden $A_1 A_2$, $B_1 B_2$ bzw. durch zwei feste Punkte P, Q, sodass

$$P A_1 A_2 \barwedge Q B_1 B_2.$$

Dann erzeugt der Schnittpunkt der entsprechenden Strahlen einen Kegelschnitt, dessen Schnittpunkte mit dem gegebenen Kreise die gesuchten Punkte sind. Somit ist unsere Aufgabe auf die fundamentale Aufgabe reduziert.

Im Falle, wo die $(2, 2)$ deutige Korrespondenz symmetrisch ist, umhüllt die Verbindungslinie der entsprechenden Punkte auf dem Kreise einen Kegelschnitt (¹). Da die Tangente dieses Kegelschnitts den Kreis im entsprechenden Punktepaare schneidet, so sind die vier Berührungspunkte der vier gemeinsamen Tangenten des Kreises sowie des Kegelschnitts die gesuchten Punkte. Damit ist unsere Aufgabe auf die reziproke Aufgabe der fundamentalen Aufgabe reduziert.

3. Jetzt sind wir in der Lage, $(1, 3)$ deutige Korrespondenz zu untersuchen. Wir denken eine $(1, 3)$ deutige Korrespondenz auf einem

---

(¹) W. K. Clifford, On the transformation of elliptic functions, London Math. Soc. 7 (1875).

Halphen, Traité des fonctions elliptiques et leurs applications, II.

Kreise, bei welcher der Punkt B dem Punkte A dreideutig entspricht. Projiziert man die entsprechenden Punkte A, B bzw. aus den Punkten O, O' des Kreises, so erzeugt der Schnittpunkt der beiden entsprechenden Strahlen eine Kurve vierter Ordnung mit dem Tripelpunkte O'. Diese Kurve hat vier weitere Punkte mit dem Kreise gemein ausser O, O', welche nichts anderes als die gesuchten Koinzidenzpunkte sind. Also reduziert sich unsere Aufgabe auf die Folgende: *Wenn eine biquadratische Kurve mit dem Tripelpunkte O' durch acht weitere Punkte O, A_i $(i=1, 2, 3, 4, 5, 6, 7)$ gegeben und ein Kegelschnitt durch die fünf Punkte O', O, B_1, B_2, B_3, gegeben sind, gesucht sei vier weitere Schnittpunkte der beiden Kurven zu finden.* Wendet man die Steinersche Transformation in Bezug auf das Dreieck A_1OO' auf die Figur an, so werden die Kurve vierter Ordnung sowie der Kegelschnitt bzw. in eine kubische Kurve mit dem Doppelpunkte O' und in einen durch die Punkte O, O' hindurchgehenden Kegelschnitt transformiert. Damit ist unser Problem auf das Folgende reduziert: *Wenn eine kubische Kurve mit dem Doppelpunkte O' durch sechs weitere Punkte gegeben und ein Kegelschnitt durch O' und vier weitere Punkte gegeben sind, gesucht sei vier weitere Schnittpunkte der beiden Kurven zu bestimmen.* Man nehme einen Punkt P auf der kubischen Kurve und verbinde P mit einem anderen Punkte Q der Kurve durch die Gerade PQ, dann bestimme den dritten Schnittpunkt R der Geraden PQ mit der Kurve (1). Wenn die Gerade PQ sich um dem Punkt P dreht, so beschreibt das Strahlenpaar $O'Q$, $O'R$ einen involutorischen Strahlenbüschel. Bezeichnet man die Schnittpunkte des Kegelschnitts mit den Strahlen $O'Q$, $O'R$ bzw. durch q, r, dann geht die Verbindungsgerade qr durch einen festen Punkt M hindurch. Also bewegt sich die Gerade Mqr zu dem Strahl PQR projektiv. Folglich erzeugt der Schnittpunkt der beiden Geraden Mqr, PQR einen Kegelschnitt. Die vier Schnittpunkte des Kegelschnitts mit dem gegebenen Kegelschnitte sind also die gesuchten Punkte.

4. Die Fundamentalaufgabe kann man mit Lineal und Zirkel lösen, wenn die vorgezeichnet gegebene rationale kubische Kurve *zirkular* ist oder die vorgezeichnet gegebene rationale biquadratische Kurve *bizirkular* ist, *viel einfacher* als F. London und ich getan haben. Es sei $\mathfrak{R}$ die

(1) Um R zu bestimmen wende man eine Steinersche Transformation in Bezug auf das Dreieck $O'PH$ auf die Figur an, wobei H einen Punkt der kubischen Kurve bedeutet. Dann werden die Kurve und die Gerade PQ bzw. in einen Kegelschnitt und eine Gerade transformiert. In der transformierten Figur kann man den Transformierten von R bestimmen.

vorgezeichnet gegebene rationale bizirkulare Kurve vierter Ordnung mit dem reellen Doppelpunkt O.

Die Fundamentalaufgabe ist unmittelbar auf die Folgende reduziert: Durch zwei Systeme von Punktquadrupeln sowie durch einen gemeinschaftlichen Punkt sind die beiden Kegelschnitte $\mathfrak{P}$, $\mathfrak{Q}$ gegeben. Man soll drei andere Schnittpunkte bestimmen.

Wendet man die Transformation der reziproken Radien in Bezug auf den Punkt O auf die Kurve $\mathfrak{R}$ an, so erhält man einen Kegelschnitt $\mathfrak{P}'$. Man denke sich nun eine Kollineation bei welcher der Kegelschnitt $\mathfrak{P}$ dem Kegelschnitte $\mathfrak{P}'$ entspricht. Es sei $\mathfrak{Q}'$ ein Kegelschnitt, welcher dem Kegelschnitt $\mathfrak{Q}$ bei dieser Kollineation entspricht. Dann entsprechen vier Schnittpunkte von $\mathfrak{P}'$, $\mathfrak{Q}'$ den vier Schnittpunkten von $\mathfrak{P}$, $\mathfrak{Q}$ von denen ein Schnittpunkt schon bekannt ist. Nun seien X_1, X_2, X_3 die drei unbekannten Schnittpunkte von $\mathfrak{P}'$, $\mathfrak{Q}'$. Nun nehme man zwei nicht-senkrechte non-parallele Geraden h, k an; durch einen Punkt O der Geraden h ziehe man zwei Geradenpaare m_1, m_2; m_1', m_2' sodass

$$\widehat{m_1 h} = \widehat{h m_2}, \qquad \widehat{m_1' h} = \widehat{h m_2'},$$

auch ziehe man zwei Geradenpaare n_1, n_2; n_1', n_2' durch einen Punkt O' der Geraden k sodass

$$\widehat{n_1 k} = \widehat{k n_2}, \qquad \widehat{n_1' k} = \widehat{k n_2'}.$$

Es seien M_1, M_2, N_1, N_2 bzw. unendlich ferne Punkte der Geraden m_1, m_2, n_1, n_2 und bestimme man die Kegelschnitte $(X_1', X_2', X_3', M_1, M_2)$, $(X_1', X_2', X_3', M_1', M_2')$, $(X_1', X_2', X_3', N_1, N_2)$, $(X_1', X_2', X_3', N_1', N_2')$. Folglich kann man leicht einen Kegelschnitt, der den beiden Kegelschnittbüscheln

$$(X_1' X_2' X_3' M_1 M_2),\ (X_1' X_2' X_3' M_1' M_2')$$

und

$$(X_1' X_2' X_3' N_1 N_2),\ (X_1' X_2' X_3' N_1' N_2')$$

gleichzeitig angehören, bestimmen ([1]). Dann ergibt sich leicht, dass dieser Kegelschnitt der durch $X_1' X_2' X_3'$ hindurchgehende Kreis ist. Diesem Kreis entspricht durch die Tranformation der reziproken Radien ein Kreis, dessen Schnittpunkte mit $\mathfrak{R}$ vier Punkte im Endlichen sind; unter diesen vier Punkten sind drei inverse Punkte X_1', X_2', X_3' enthalten und folglich sind X_1, X_2, X_3 gefunden. Damit ist die Bestimmung der Punkte X_1, X_2, X_3 erledigt. Den Fall, wo eine vorgezeichnet gegebene kubische Kurve zirkular ist, kann man in gleicher Weise behandeln.

Sendai, den 25 ten Februar, 1918.

([1]) T. Kubota, a. a. O. S. 30.

Bemerkung zur Theorie der Approximation der irrationalen Zahlen durch rationale Zahlen,

von

Matsusaburô Fujiwara in Sendai.

In meiner vorigen gleichnamigen Arbeit(¹) habe ich eine Methode(²) angegeben, durch welche der Hurwitzsche Satz mit seinen Ergänzungen in einem Schlag bewiesen wird. Ich werde hier nochmals dieselbe Methode zur Anwendung bringen, um den Vahlenschen Satz und seine Verallgemeinerung herzuleiten.

Es sei ω irgend eine irrationale Zahl und sei $\frac{P_n}{Q_n}$ ihr n^{ter} Näherungsbruch in der Kettenbruchentwicklung

$$\omega = a_0 + \frac{1}{a_1+}\ \frac{1}{a_2+}\ \cdots\cdots.$$

Dann ist

$$\omega - \frac{P_n}{Q_n} = \frac{(-1)^{n-1}}{Q_n(\omega_{n+1}\, Q_n + Q_{n-1})}, \qquad (1)$$

worin

$$\omega_{n+1} = a_{n+1} + \frac{1}{\omega_{n+2}}$$

bedeutet.

Nun setze man voraus, dass die zwei aufeinanderfolgenden Näherungsbrüche $\frac{P_n}{Q_n}$ und $\frac{P_{n+1}}{Q_{n+1}}$ die Ungleichungen

$$\left|\omega - \frac{P_n}{Q_n}\right| \geqq \frac{1}{\lambda\, Q_n^2}, \quad \left|\omega - \frac{P_{n+1}}{Q_{n+1}}\right| \geqq \frac{1}{\lambda\, Q_{n+1}^2} \qquad (2)$$

(¹) Dieses Journal, 13, Juni 1917. Ich benütze diese Gelegenheit einige Fehler dort zu berichtigen: es soll das Suffix von ω, a um 1 vermehrt werden; man liest insbesondere a_{n+1} anstatt a_n in dem Satz IV.

(²) Diese Methode wurde auch von Herrn Prof. Humbert in seiner Arbeit, Remarques sur certaines suites d'approximations, Liouville Journ., (7) 2, Fasc. 3, 1916 veröffentlicht; dieses Heft ist zu mir aber erst im August 1917 gelangt.

erfüllen. Daraus werden wir eine wichtige Folgerung für a_{n+1} und a_{n+2} ableiten.

Aus den Relationen (1), (2) haben wir

$$\lambda \geqq \omega_{n+1} + \frac{Q_{n-1}}{Q_n}, \tag{3}$$

$$\lambda \geqq \omega_{n+2} + \frac{Q_n}{Q_{n+1}}, \tag{4}$$

und wenn wir die Relationen

$$\omega_{n+1} = a_{n+1} + \frac{1}{\omega_{n+2}},$$

$$\frac{Q_{n+1}}{Q_n} = a_{n+1} + \frac{Q_{n-1}}{Q_n}$$

berücksichtigen, so bekommen wir

$$\omega_{n+1} + \frac{Q_{n-1}}{Q_n} = \frac{1}{\omega_{n+2}} + \frac{Q_{n+1}}{Q_n},$$

folglich

$$\lambda \geqq \frac{1}{\omega_{n+2}} + \frac{Q_{n+1}}{Q_n}. \tag{5}$$

Nach (4) und (5) hat man

$$(\lambda - \omega_{n+2})\left(\lambda - \frac{1}{\omega_{n+2}}\right) \geqq \frac{Q_n}{Q_{n+1}} \cdot \frac{Q_{n+1}}{Q_n} = 1,$$

$$\left(\lambda - \frac{Q_{n+1}}{Q_n}\right)\left(\lambda - \frac{Q_n}{Q_{n+1}}\right) \geqq \omega_{n+2} \cdot \frac{1}{\omega_{n+2}} = 1,$$

d.h.

$$\lambda \geqq \omega_{n+2} + \frac{1}{\omega_{n+2}}, \tag{6}$$

$$\lambda \geqq \frac{Q_{n+1}}{Q_n} + \frac{Q_n}{Q_{n+1}}. \tag{7}$$

Man erhält daraus weiter

$$\frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \leqq \omega_{n+2} \leqq \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}, \tag{8}$$

$$\frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \leqq \frac{Q_{n+1}}{Q_n} \leqq \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}, \tag{9}$$

welche in unserer Untersuchung grundlegend ist.

Wenn man hier die Tatsachen

$$\omega_{n+2}=a_{n+2}+\frac{1}{\omega_{n+3}}>a_{n+2},$$

und

$$\frac{Q_{n+1}}{Q_n}=a_{n+1}+\frac{Q_{n-1}}{Q_n}>a_{n+1}$$

in Betracht zieht, so folgert man daraus

$$a_{n+2}<\frac{\lambda+\sqrt{\lambda^2-4}}{2}, \tag{10}$$

bzw.

$$a_{n+1}<\frac{\lambda+\sqrt{\lambda^2-4}}{2}. \tag{11}$$

Da a_{n+1}, $a_{n+2}\geqq 1$ ist, schliesst man aus (10), (11)

$$\frac{\lambda+\sqrt{\lambda^2-4}}{2}>1, \quad \text{i.e.} \quad \lambda>2.$$

Also erhält man den Satz von Vahlen[1]:

Mindestens einer von zwei aufeinanderfolgenden Näherungsbrüchen irgend einer irrationalen Zahl ω genügt der Ungleichung

$$\left|\omega-\frac{P_n}{Q_n}\right|<\frac{1}{2Q_n^2}.$$

Setzt man zunächst in (10), (11)

$$\frac{\lambda+\sqrt{\lambda^2-4}}{2}=2, \quad \text{i.e.} \quad \lambda=\frac{5}{2},$$

so ergibt sich

$$a_{n+2}<2, \quad \text{bzw.} \quad a_{n+1}<2.$$

Daraus schliesst man, dass, wenn $a_{n+1}<2$, oder $a_{n+2}<2$ ist, mindestens eine der Ungleichungen

$$\left|\omega-\frac{P_n}{Q_n}\right|<\frac{1}{\frac{5}{2}Q_n^2}, \quad \left|\omega-\frac{P_{n+1}}{Q_{n+1}}\right|<\frac{1}{\frac{5}{2}Q_{n+1}^2}$$

---

(1) Vahlen: Über Näherungswerthe und Kettenbrüche, Journ. für Math. 115, 1895.

gilt. Also kann man den folgenden Satz aussagen.

Ist $a_{n+1} \geqq 2$, *so ist entweder*

$$\left| \omega - \frac{P_n}{Q_n} \right| < \frac{1}{\frac{5}{2} Q_n^2},$$

oder

$$\left| \omega - \frac{P_{n-1}}{Q_{n-1}} \right| < \frac{1}{\frac{5}{2} Q_{n-1}^2}, \quad \left| \omega - \frac{P_{n+1}}{Q_{n+1}} \right| < \frac{1}{\frac{5}{2} Q_{n+1}^2}.$$

Wenn man ausser (1) noch

$$\left| \omega - \frac{P_{n-1}}{Q_{n-1}} \right| \geqq \frac{1}{\lambda Q_{n-1}^2}$$

voraussetzt, so erhält man ausser (8), (9) weiter

$$\frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \leqq \omega_{n+1} \leqq \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}, \tag{8'}$$

$$\frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \leqq \frac{Q_n}{Q_{n-1}} \leqq \frac{\lambda + \sqrt{\lambda^2 - 4}}{2}. \tag{9'}$$

Aus (8) und (8′) oder aus (9) und (9′) ergibt sich

$$a_{n+1} + \frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \leqq \frac{\lambda + \sqrt{\lambda^2 - 4}}{2},$$

d.h.

$$a_{n+1} \leqq \sqrt{\lambda^2 - 4}, \tag{12}$$

wo das Gleichheitszeichen ausgeschlossen ist, wenn $\sqrt{\lambda^2 - 4}$ irrational ist. Daraus schliesst man unmittelbar, wie ich schon in der vorigen Arbeit gezeigt habe, den Hurwitzschen Satz und seine Ergänzungen:

Mindestens einer von drei aufeinanderfolgenden Näherungsbrüchen irgend einer irrationalen Zahl ω *genügt der Ungleichung*

$$\left| \omega - \frac{P_n}{Q_n} \right| < \frac{1}{\sqrt{5}\, Q_n^2}.$$

Ist $a_{n+1} \geqq 2$, *so gilt mindestens eine von den folgenden Ungleichungen*

$$\left| \omega - \frac{P_\nu}{Q_\nu} \right| < \frac{1}{\sqrt{8}\, Q_\nu^2}, \quad (\nu = n-1,\ n,\ n+1);$$

in der Folge gibt es unendlichviele Näherungsbrüche $\dfrac{P_n}{Q_n}$ *irgend einer mit* $\dfrac{1+\sqrt{5}}{2}$ *nicht äquivalenten irrationalen Zahl* ω, *für welche*

$$\left|\omega-\frac{P_n}{Q_n}\right|<\frac{1}{\sqrt{8}\,Q_n^2}$$

gilt.

Damit ist gezeigt, dass alle diese Sätze aus den Relationen (8) und (9) als unmittelbare Folge abgeleitet werden.

Zum Schluss werden wir die oben angegebene Methode auf den Fall der Kettenbruchentwicklung einer komplexen irrationalen Zahl anwenden.

Wir legen einen beliebigen Zahlkörper K zu Grund und betrachten irgend eine Kettenbruchentwicklung einer irrationalen Zahl ω in K, worin die folgenden Bedingungen erfüllt sind:

(1) $|Q_n|>|Q_{n-1}|$;

(2) es existiert eine positive Zahl $k(>1)$, von der Art $|\omega_n|\geqq k$, wo P_n, Q_n und $a_0, a_1, a_2, \cdots\cdots$ die ganze Zahlen im Körper K bedeuten.

Wie vorhin setze man voraus, dass

$$\left|\omega-\frac{P_n}{Q_n}\right|\geqq\frac{1}{\lambda|Q_n|^2},\quad \left|\omega-\frac{P_{n+1}}{Q_{n+1}}\right|\geqq\frac{1}{\lambda|Q_{n+1}|^2};$$

wenn man

$$\omega-\frac{P_\nu}{Q_\nu}=\frac{1}{\delta_\nu\,Q_\nu^2}$$

setzt, so ergibt sich

$$\lambda\geqq|\delta_n|,\ |\delta_{n+1}|,\quad |\delta_\nu|\geqq|\omega_{\nu+1}|-\left|\frac{Q_{\nu-1}}{Q_\nu}\right|\geqq k-1,$$

$$\delta_n=(-1)^n\left(\frac{1}{\omega_{n+2}}+\frac{Q_{n+1}}{Q_n}\right),\quad \delta_{n+1}=(-1)^{n+1}\left(\omega_{n+2}+\frac{Q_n}{Q_{n+1}}\right),$$

woraus bekommt man sofort

$$\left((-1)^n\delta_n-\frac{1}{\omega_{n+2}}\right)\left((-1)^{n+1}\delta_{n+1}-\omega_{n+2}\right)=1,$$

d.h.

$$(-1)^{n+1}=\frac{\omega_{n+2}}{\delta_{n+1}}-\frac{1}{\omega_{n+2}\,\delta_n},$$

und ferner

$$1 \geqq \frac{|\omega_{n+2}|}{\lambda} - \frac{1}{(k-1)|\omega_{n+2}|}.$$

In der Folge

$$|\omega_{n+2}| \leqq \left(\lambda + \sqrt{\lambda^2 + \frac{4\lambda}{k-1}}\right) : 2,$$

und aus $|\omega_{n+2}| \geqq k$

$$k \leqq \left(\lambda + \sqrt{\lambda^2 + \frac{4\lambda}{k-1}}\right) : 2, \quad \text{d.h.} \quad \lambda \geqq \frac{k^2(k-1)}{1+k(k-1)}.$$

Also: *Ist* $\lambda_0 = \dfrac{k^2(k-1)}{1+k(k-1)}$, *dann genügt mindestens einer der zwei aufeinanderfolgenden Näherungsbrüche irgend einer komplexen irrationalen Zahl* ω *der Ungleichung*

$$\left|\omega - \frac{P_n}{Q_n}\right| < \frac{1}{\lambda |Q_n|^2}, \quad (\lambda < \lambda_0).$$

Wenn man von der Voraussetzung

$$\left|\omega - \frac{P_n}{Q_n}\right| > \frac{1}{\lambda |Q_n|^2}, \quad \left|\omega - \frac{P_{n+1}}{Q_{n+1}}\right| > \frac{1}{\lambda |Q_{n+1}|^2}$$

ausgeht, so kann man den eben bewiesenen Satz in der folgenden Form ausdrücken:

Mindestens einer von zwei aufeinanderfolgenden Näherungsbrüchen irgend einer komplexen irrationalen Zahl ω *genüyt der Ungleichung*

$$\left|\omega - \frac{P_n}{Q_n}\right| \leqq \frac{1}{\lambda_0 |Q_n|^2}.$$

Wenn wir insbesondere die Kettenbruchentwicklung im Körper $K(i)$ in Betracht ziehen, welche zuerst von Herrn Prof. Hurwitz[1] behandelt wurde, so haben wir

$$k = \sqrt{2}, \quad \lambda_0 = \frac{2(2\sqrt{2}-1)}{7} > \frac{1}{2}.$$

Also: *Mindestens einer von zwei aufeinanderfolgenden Näherungsbrüchen irgend einer komplexen irrationalen Zahl im Körper* $K(i)$ *genügt der Ungleichung*

---

(1) A. Hurwitz: Über die Entwicklung complexer Grössen in Kettenbrüche. Acta Math., 11, 1887–88.

$$\left| \omega - \frac{P_n}{Q_n} \right| \leqq \frac{1}{\lambda_0 |Q_n|^2}, \qquad \lambda_0 = \frac{2}{7}(2\sqrt{2}-1);$$

insbesondere der Ungleichung

$$\left| \omega - \frac{P_n}{Q_n} \right| < \frac{2}{|Q_n|^2}.$$

Dies kann man als eine Verallgemeinerung des Vahlenschen Satzes betrachten.

Sendai, März 1918.

The Stability of the Parachute,

by

S. Brodetsky, Bristol, England.

1. A parachute can be defined mathematically as a body symmetrical about an axis, descending under the influence of gravity and the resistance of the air, with the axis of symmetry vertical. If the parachute suffers any disturbance, the stability or instability of the oscillations set up will depend upon the extent and form of the "umbrella" experiencing the air pressure, and the disposition of the masses constituting the parachute and passenger.

The undisturbed motion of a body falling vertically in a resisting medium is discussed in any standard treatise on dynamics. There is a certain velocity, called the terminal velocity, which represents the maximum speed attainable in such motion. In practice this velocity is acquired fairly soon, and therefore it will be sufficient to consider the stability of a parachute descending with the terminal velocity.

We take the centre of gravity of the parachute and passenger combined as the origin of rectangular coordinates. The y axis is taken along the downward direction of the axis of symmetry, and two lines at right angles to one another in a perpendicular plane through the centre of gravity are taken as the axes of x and z respectively. The axes thus defined are fixed in the body and move with it.

In the undisturbed motion the y axis is vertically downwards, and axes of x and z are horizontal. To obtain the directions of the axes in space in a disturbed position of the parachute, we follow the notation used by Bryan in *Stability in Aviation*. We suppose the parachute rotated first about the axis of y (from z to x) through an angle ψ, then about the axis of z (from x to y) through an angle θ, and finally about the axis of x (from y to z) through an angle φ. Then the cosines of the angles that the new directions of the axes make with the downward vertical are

$$\sin\theta, \quad \cos\theta\cos\varphi, \quad -\cos\theta\sin\varphi.$$

Let W be the total weight of the parachute and passenger, the

latter being supposed to hold on tight so that he has no independent oscillation; A, B, A the moments of inertia (in gravitational units) about the axes of x, y and z respectively. The products of inertia are all zero by symmetry. Let V be the terminal velocity. If u, $V+v$, w are the velocities of the centre of gravity along the axes of x, y, z, and p, q, r are the angular velocities of the parachute about these axes, in the disturbed motion, then the equations of motion are

$$\left.\begin{aligned}
&\frac{W}{g}\left(\frac{du}{dt}+qw-r(V+v)\right)=W\sin\theta-X,\\
&\frac{W}{g}\left(\frac{dv}{dt}+ru-pw\right)=W\cos\theta\cos\varphi-Y,\\
&\frac{W}{g}\left(\frac{dw}{dt}+p(V+v)-qu\right)=-W\cos\theta\sin\varphi-Z,\\
&\frac{A}{g}\frac{dp}{dt}+\frac{A-B}{g}rq=-L,\\
&\frac{B}{g}\frac{dq}{dt}=-M,\\
&\frac{A}{g}\frac{dr}{dt}+\frac{B-A}{g}pq=-N,
\end{aligned}\right\}\qquad(1)$$

where X, Y, Z, L, M, N are the components of force and couple representing the air resistance, reckoned positive when they tend to oppose the respective components of motion.

2. To investigate the stability we suppose u, v, w, p, q, r and also θ and φ to be very small, and neglect squares and products of small quantities. The equations of motion become

$$\left.\begin{aligned}
&\frac{W}{g}\left(\frac{du}{dt}-Vr\right)=W\theta-X, &&\frac{A}{g}\frac{dp}{dt}=-L,\\
&\frac{W}{g}\frac{dv}{dt}=W-Y, &&\frac{B}{g}\frac{dq}{dt}=-M,\\
&\frac{W}{g}\left(\frac{dw}{dt}+Vp\right)=-W\varphi-Z, &&\frac{A}{g}\frac{dr}{dt}=-N.
\end{aligned}\right\}\qquad(2)$$

In the undisturbed motion we have

$$X_0=Z_0=L_0=M_0=N_0=0,\quad Y_0=W.$$

We therefore write

$$X = u\,X_u + v\,X_v + w\,X_w + p\,X_p + q\,X_q + r\,X_r,$$

with similar expressions for Z, L, M, N; also

$$Y = Y_0 + u\,Y_u + v\,Y_v + w\,Y_w + p\,Y_p + q\,Y_q + r\,Y_r.$$

We suppose the air to be quiet.

Because of the symmetry about the y axis, and therefore also about the xy and yz planes, it can be at once seen that all the resistance derivatives are zero except

$$X_u,\ X_r,\ Y_v,\ Z_w,\ Z_p,\ L_w,\ L_p,\ M_q,\ N_u,\ N_r\,.$$

The equations of motion thus simplify into

$$\left.\begin{aligned}
\frac{W}{g}\frac{du}{dt}-\frac{WV}{g}r &= W\theta - u\,X_u - r\,X_r,\\
\frac{W}{g}\frac{dv}{dt} &= -v\,Y_v,\\
\frac{W}{g}\frac{dw}{dt}+\frac{WV}{g}p &= -W\varphi - w\,Z_w - p\,Z_p,\\
\frac{A}{g}\frac{dp}{dt} &= -w\,L_w - p\,L_p,\\
\frac{B}{g}\frac{dq}{dt} &= -q\,M_q,\\
\frac{A}{g}\frac{dr}{dt} &= -u\,N_u - r\,N_r.
\end{aligned}\right\} \qquad (3)$$

By virtue of the relations

$$p = \frac{d\varphi}{dt} + \sin\theta\,\frac{d\psi}{dt}\,, \quad r = \cos\varphi\,\frac{d\theta}{dt} - \cos\theta\sin\varphi\,\frac{d\psi}{dt}\,,$$

we can use

$$\frac{d\varphi}{dt} = p\,, \quad \frac{d\theta}{dt} = r.$$

Hence the equations (3) can be separated as follows:

$$\frac{W}{g}\frac{dv}{dt} = -v\,Y_v \qquad (3_v)$$

by itself;

$$\frac{B}{g}\frac{dq}{dt} = -q\,M_q \qquad (3_q)$$

by itself; the pair of equations

$$\left.\begin{aligned}\frac{W}{g}\frac{d\,u}{d\,t}-\frac{W\,V}{g}\,r&=W\theta-u\,X_u-r\,X_r,\\ \frac{A}{g}\frac{d\,r}{d\,t}\qquad\quad&=\qquad -u\,N_u-r\,N_r,\end{aligned}\right\}\qquad(3_{u,\,r})$$

and the pair of equations

$$\left.\begin{aligned}\frac{W}{g}\frac{d\,w}{d\,t}+\frac{W\,V}{g}p&=-W\varphi-w\,Z_w-p\,Z_p,\\ \frac{A}{g}\frac{d\,p}{d\,t}\qquad\quad&=\qquad -w\,L_w-p\,L_p.\end{aligned}\right\}\qquad(3_{w,\,p})$$

3. The equation (3_v) represents the oscillation in the vertical motion about the terminal velocity. Since Y_v is obviously positive, it follows that this oscillation dies out, and is therefore stable.

The equation (3_q) represents the rotation of the parachute about the axis of symmetry, i. e. about the vertical practically. M_q is also positive, owing to air friction, so that this rotation, if generated, dies out.

The equations $(3_{u,\,r})$ give the translation and rotation in the plane $x\,y$; similarly the equations $(3_{w,\,p})$ give the translation and rotation in the plane $y\,z$. Since these are separate pairs of equations, it follows that the motions in the planes $x\,y$ and $y\,z$ can be treated separately. Thus the general theory of the stability of the parachute is only a two-dimensional problem.

In $(3_{u,\,r})$ put

$$\frac{d\,u}{d\,t}=\lambda\,u,\quad \frac{d\,r}{d\,t}=\lambda\,r,\quad \frac{d\,\theta}{d\,t}=\lambda\,\theta,$$

where λ is the exponent in the exponential function representing the oscillation. Then we write

$$\theta=\frac{1}{\lambda}\frac{d\,\theta}{d\,t}=\frac{r}{\lambda},$$

and we get for λ the two equations

$$\left.\begin{aligned}\frac{W}{g}\lambda\,u-\frac{W\,V}{g}r&=\frac{W\,r}{\lambda}-u\,X_u-r\,X_r,\\ \frac{A}{g}\lambda\,r\qquad\quad&=\qquad -u\,N_u-r\,N_r.\end{aligned}\right\}\qquad(4_{u,\,r})$$

Eliminating the ratio $u\,/\,r$ we get for λ the determinantal equation

$$\begin{vmatrix} \frac{W}{g}\lambda + X_u & -\frac{WV}{g} - \frac{W}{\lambda} + X_r \\ N_u & \frac{A}{g}\lambda + N_r \end{vmatrix} = 0 . \qquad (5_{u,r})$$

Similarly the oscillation in the yz plane is governed by the determinantal equation

$$\begin{vmatrix} \frac{W}{g}\lambda + Z_w & \frac{WV}{g} + \frac{W}{\lambda} + Z_p \\ L_w & \frac{A}{g}\lambda + L_p \end{vmatrix} = 0 . \qquad (5_{w,p})$$

As a check on our results we can use the fact that by symmetry the equations $(5_{u,r})$ and $(5_{w,p})$ should be identical. This is so, since the rotation in the yz plane is opposite in sense to that in the xy plane, and therefore by symmetry

$$Z_w = X_u, \quad L_w = -N_u, \quad Z_p = -X_r, \quad L_p = N_r.$$

We shall therefore consider only the equation $(5_{u,r})$, and the stability or instability of the parachute will be determined by the signs of the real parts of the solutions for λ of this equation.

4. The surface whose air-resistance is most important in the parachute is umbrella-shaped and not quite but nearly flat. As a first approximation we may consider it flat and ignore the skin friction. Thus we can put

$$X_u = X_r = 0 .$$

The equation for λ is

$$\frac{WA}{g^2}\lambda^3 + \frac{WN_r}{g}\lambda^2 + \frac{WVN_u}{g}\lambda + WN_u = 0 . \qquad (6)$$

Routh's conditions for stability (Advanced Rigid Dynamics, p. 228) are that

$$N_u, \ N_r, \ N_u(VN_r - A)$$

shall all be positive. It is clear that N_u is always positive, since it is well known by experiment that a positive velocity u gives a centre of pressure displaced in the positive direction of the x axis. The conditions for stability thus become

$$N_r > 0 , \quad VN_r - A > 0 ,$$

and clearly both are included in the second.

We thus conclude that the parachute is stable if

$$V N_r > A. \qquad (7)$$

To find N_r for any lamina we use the fact that air resistances are proportional to the squares of the rates of displacement. Hence if the centre of gravity is at a distance d below the lamina, we use

$$N = V^2 f\left(\frac{d\,r}{V}, \frac{r}{V}\right),$$

dr/V representing the complement of the angle of attack, and r/V the ratio of the angular velocity to the velocity of translation. It can, however, be shown that to the first order of small quantities the second argument does not appear in N. (For this proof I am indebted to a suggestion by Prof. Bryan.) For suppose that d is zero; then $N = f(0, r/V)$. If now V is kept the same in value but is reversed in sign, it is evident that N must retain its original value and its original sign. Thus $f(0, r/V)$ contains only even powers of r/V, so that when r is small we can consider $f(0, r/V)$ to be constant, clearly zero. Thus to the first order we can take

$$N = V^2 \phi\left(\frac{d\,r}{V}\right),$$

where ϕ is some functional form. Neglecting squares and higher powers of r, and remembering that N is zero for r zero, we get

$$N = c\,d\,r\,V, \qquad (8)$$

where c is a constant depending on the air density, the coefficient of air resistance, and the dimensions of the parachute. We have

$$N_r = c\,d\,V,$$

and the condition of stability is

$$c\,d\,V^2 > A. \qquad (9)$$

Now

$$W = e\,V^2, \qquad (10)$$

where e is another constant like c. Hence we get for (8) the condition

$$\frac{c\,d}{e} > k^2, \qquad (11)$$

where k is the radius of gyration of the parachute about a horizontal axis through the centre of gravity.

The quantity c/e is a constant from which the air density and the coefficient of air resistance have been eliminated. Also from considerations of dimensions it follows that c/e is proportional to the radius of the umbrella. Call this a. Then the condition is

$$a\,d > \rho\, k^2, \tag{12}$$

where ρ is a constant depending only upon the fact that the umbrella is a circular lamina.

From (8) and (10) it is readily deducible that if the normal pressure and the couple due to the air resistance for a velocity U at an angle of attack α are respectively

$$K\,S\,U^2 f_1(\alpha)\,, \quad K\,S\,U^2\,a\,f_2(\alpha)\,,$$

then

$$\rho = -\left\{ f_1(\alpha) \Big/ \frac{d}{d\,\alpha} f_2(\alpha) \right\}_{\alpha=\frac{\pi}{2}}, \tag{13}$$

ρ being in fact a positive quantity. It is of interest that the stability is independent of the nature of the resisting medium, as e. g. its density.

5. The quantity ρ seems to be unobtainable for a circular plate, owing to the lack of experimental data for a lamina of this form. For a square plate with rotation about a bisector of the angle between the diameters, ρ can be calculated by means of the empirical formulae obtained by various experimenters. Using Buchemin's and Soreau's result we have

$$f_1(\alpha) = \frac{2\sin\alpha}{1+\sin^2\alpha}\,, \quad f_2(\alpha) = \frac{\sin\alpha\cos\alpha}{(1+\sin^2\alpha)(\cos\alpha + 2\sin\alpha)}\,.$$

We find $\rho = 4$.

For a circular plate ρ is probably not very different from 4, but the matter needs experimental investigation.

6. The simplest form of the condition (12) is

$$a > \rho \frac{k^2}{d} \tag{14}$$

giving a lower limit to the radius of the umbrella for given masses, etc.

If the radius a is given and we wish to find the possible disposition

of the masses in the parachute so as to ensure stability, we proceed as follows.

Let the umbrella and appendages have a weight W_1 with centre of gravity at depth d_1 below the surface of the umbrella, and let the weight of the passenger be W_2 with centre of gravity at depth d_2. Let $W_1 k_1^2$ be the moment of inertia of the former about an axis through its centre of gravity parallel to the plane of the umbrella, and $W_2 k_2^2$ the moment of inertia of the latter about a parallel axis through its centre of gravity. Then we have

$$d=\frac{W_1 d_1+W_2 d_2}{W_1+W_2},$$

$$k^2=\frac{W_1(k_1^2+(d-d_1)^2)+W_2(k_2^2+(d_2-d)^2)}{W_1+W_2}.$$

Thus the condition of stability is

$$\frac{m(1-m)D^2+(1-m)k_1^2+m k_2^2}{mD+d_1}<\frac{a}{\rho},$$

where $m=W_2/(W_1+W_2)$ so that $W_1/(W_1+W_2)=1-m$, and D is the depth of the centre of gravity of the passenger below the centre of gravity of the parachute itself. The condition can be written

$$m(1-m)D^2-\frac{ma}{\rho}D+(1-m)k_1^2+m k_2^2-\frac{a}{\rho}d_1<0. \quad (15)$$

Since $m(1-m)$ is positive, the condition is that the equation

$$m(1-m)D^2-\frac{ma}{\rho}D+(1-m)k_1^2+m k_2^2-\frac{a}{\rho}d_1=0 \quad (16)$$

shall have real roots and that D shall lie between these roots.

Hence stability is possible if

$$\left(\frac{a}{\rho}\right)^2>4\frac{1-m}{m}\left\{(1-m)k_1^2+m k_2^2-\frac{a}{\rho}d_1\right\} \quad (17)$$

and it is ensured if D lies between the values

$$\frac{1}{2(1-m)}\left\{\frac{a}{\rho}\pm\left[\left(\frac{a}{\rho}\right)^2-4\frac{1-m}{m}\left((1-m)k_1^2+m k_2^2-\frac{a}{\rho}d_1\right)\right]^{1/2}\right\}. \quad (18)$$

We see that the parachute descent is safe if the passenger holds on at a depth lying between two limits, assuming that the condition (17) is already satisfied.

A Generalized Pascal Theorem on a Space Cubic,

by

KINNOSUKE OGURA, Ôsaka.

In this note I will give a *synthetic* proof of the following theorem on a space cubic, which may be considered as *a generalization*(1) *of the Pascal theorem on a conic:*

When $1, 2, 3, \cdots\cdots, 12$ *are any twelve points on a space cubic, the four points*

$$(\mathrm{I})\qquad \begin{cases} [(1,2,3),\ (5,6,\ 7),\ (\ 9,10,11)], \\ [(2,3,4),\ (6,7,\ 8),\ (10,11,12)], \\ [(3,4,5),\ (7,8,\ 9),\ (11,12,\ 1)], \\ [(4,5,6),\ (8,9,10),\ (12,\ 1,\ 2)] \end{cases}$$

are in a plane(2).

---

(1) For other generalizations concerning a space cubic, see Encyclopédie des sciences mathématiques, (3) 4, fasc. 1 (1914), p. 128.

(2) Let θ_i $(i=1, 2, \cdots, 12)$ be any given quantities and $f_{i,j,k}$ denote the binary cubic forms $(x-\theta_i\, y)(x-\theta_j\, y)(x-\theta_k\, y)$. Then the theorem is equivalent to any one of the following two *algebraic* theorems:

I. *There exist the twelve constants* $\lambda, \mu, \nu, \cdots, \nu'''$, *for which we have the identities*

$$\lambda f_{1,2,3}+\mu f_{5,6,7}+\nu f_{9,10,11}\equiv\lambda' f_{2,3,4}+\mu' f_{6,7,8}+\nu' f_{10,11,12}$$
$$\equiv\lambda'' f_{3,4,5}+\mu'' f_{7,8,9}+\nu'' f_{11,12,1}\equiv\lambda''' f_{4,5,6}+\mu''' f_{8,9,10}+\nu''' f_{12,1,2}.$$

For the case of a conic, see Laguerre, Sur la représentation des formes binaires dans le plan et dans l'espace, Bull. de la Soc. Philomatique, (1) 40 (1872), p. 221 [=Oeuvres, II, p. 277].

II. *There exist the four constants* k_1, k_2, k_3, k_4, *for which we have the identity*

$$k_1\, K(f_{1,2,3};\, f_{5,6,7};\, f_{9,10,11})+k_2\, K(f_{2,3,4};\, f_{6,7,8};\, f_{10,11,12})$$
$$+k_3\, K(f_{3,4,5};\, f_{7,8,9};\, f_{11,12,1})+k_4\, K(f_{4,5,6};\, f_{8,9,10};\, f_{12,1,2})\equiv 0,$$

$K(f, \varphi, \psi)$ *standing for the determinant*

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 \varphi}{\partial x^2} & \dfrac{\partial^2 \psi}{\partial x^2} \\[2ex] \dfrac{\partial^2 f}{\partial x\, \partial y} & \dfrac{\partial^2 \varphi}{\partial x\, \partial y} & \dfrac{\partial^2 \psi}{\partial x\, \partial y} \\[2ex] \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 \varphi}{\partial y^2} & \dfrac{\partial^2 \psi}{\partial y^2} \end{vmatrix},$$

Let us suppose that $2, 3, 4, \cdots\cdots, 12$ are eleven fixed points on a given space cubic; and consider the correspondence between the points 1 and $\bar{1}$, for which the four points

$$\text{(II)} \quad \begin{cases} A \equiv [(1,2,3),\ (5,6,\ 7),\ (\ 9,10,11)], \\ B \equiv [(2,3,4),\ (6,7,\ 8),\ (10,11,12)], \\ C \equiv [(3,4,5),\ (7,8,\ 9),\ (11,12,\ \bar{1})], \\ D \equiv [(4,5,6),\ (8,9,10),\ (12,\ \bar{1},\ 2)] \end{cases}$$

are in a plane.

When 1 is given, the two points A and B are fixed and lie on the fixed line $AB(\equiv l)$. Also C is on the fixed line

$$m \equiv \{(3,4,5),\ (7,8,9)\},$$

and D on the fixed line

$$n \equiv \{(4,5,6),\ (8,9,10)\};$$

and the line CD intersects the line l. Let us take any point $1'$ on the cubic, and let C' be the point of intersection of m and the plane $(11, 12, 1')$, and D' be that of n and the line cutting l and n and passing through C'. If $1''$ be the third point of intersection of the cubic and the plane $(12, D', 2)$, then there exists a one-one correspondence between $1'$ and $1''$; so that there are two self-corresponding points (that is, the points $\bar{1}$, for which the four points (II) are in a plane). Denote these points by $\bar{1}_1$ and $\bar{1}_2$.

Conversely, when $\bar{1}_1$ is given the three points B, C, D are determined uniquely. Hence the plane, passing through 2, 3 and the point of intersection of the plane (B, C, D) and the line $\{(5,6,7),\ (9,10,11)\}$, cuts the cubic at the third point 1. When $\bar{1}_2$ is given, a similar result will be obtained.

It follows that we have a one-two correspondence between 1 and $\bar{1}$; so that there are three or ∞^1 self-corresponding points (that is, the points 1, for which the four points (I) are in a plane).

But we can prove that *the locus of the points 1 in space, for which*

---

which was treated by Profs. Rosanes, Lindemann and Hayashi. (See Ogura, Binary forms and duality, Tôhoku Math. Journ., (1918), p. 290.)

For the case of a conic, see Hesse, Zur Involution, Crelle's Journal, 63 (1864) [=Werke, p. 515]; and Fr. Meyer, Allgemeine Formen- und Invariantentheorie, 1 (1909), p. 361, where Prof. Meyer proposed to solve the question "····Wie lautet die entsprechende Übertragung auf kubische Raumkurven?"

the four points (I) are in a plane (P), is a cubic surface. Consider the tetrahedron having the faces

$$(P)\,; \quad (1, 2, 3)\,; \quad (11, 12, 1)\,; \quad (12, 1, 2),$$

which contain

the fixed point $[(2, 3, 4),\ (6, 7, 8),\ (10, 11, 12)]$;
the fixed line $\{2,\ 3\}$;
the fixed line $\{11, 12\}$;
and the fixed line $\{12,\ 2\}$

respectively. Since the three edges

$$\{(P), (1, 2, 3)\}\,; \quad \{(P), (11, 12, 1)\}\,; \quad \{(P), (12, 1, 2)\}$$

cut the three fixed lines

$$\{(5, 6, 7),\ (9, 10, 11)\}\,;\ \{(3, 4, 5),\ 7, 8, 9)\}\,;\ \{(4, 5, 6), (8, 9, 10)\}$$

respectively, we obtain three trilinear point ranges. Hence the three faces

$$(1, 2, 3); \qquad (11, 12, 1); \qquad (12, 1, 2)$$

form three trilinear axial pencils; whence the locus of the vertex 1 is a cubic surface([1]).

Therefore we have, at least, nine points 1 on the space cubic, for which the four points (I) are in a plane; and consequently any point on the cubic can be taken as the point 1. Thus the theorem has been established.

Lastly we remark that *a similar theorem holds good for the rational curve in space of n dimensions:*

$$\rho x_1 = \theta^n, \quad \rho x_2 = \theta^{n-1}, \cdots\cdots \ \cdots\cdots, \ \rho x_n = \theta, \quad \rho x_{n+1} = 1,$$

θ being the parameter.

Ikeda near Ôsaka, March 1918.

([1]) F. August, De superficiebus tertii ordinis, Diss. Berlin 1862: F. London, Zur Theorie der trilinearen Verwandtschaft dreier einstufiger Grundgebilde, Math. Ann., 44 (1894), p. 405; R. Sturm, Die Lehre von den geometrischen Verwandtschaften, 1 (1908), p. 324.

和算中ノ術,特ニ統術ニ就テ

On Some Methods, Especially Tôjutsu, in the Japanese Mathematics,

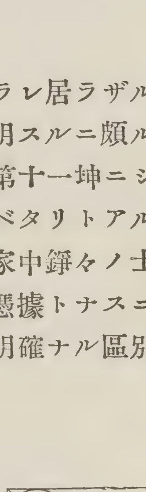
林　鶴　一（仙　臺）

Tsuruichi Hayashi, Sendai.

和算ニ於ケル諸種ノ術名ニツキテハ其ノ定義ノ揭ゲラレ居ラザルヲ常トスルヲ以テ,正當ニ某術トハ如何ナルモノナリト說明スルニ頗ル困難ヲ感ズル場合多シ. 關算完傳第九十六卷ハ算藪漫錄第十一坤ニシテ,其末ニ藤田定資ガ明和五年戊子(1768)冬至ニ於テ述ベタリトアル一題十五解アリ,各解ニ術名ヲ附ス,藤田定資ハ關流和算家中錚々ノ士タリ,依リテ以テ數種ノ術ノ如何ナルモノナルカヲ知ルノ憑據トナスニ足ラン. 以下其說述ニ聊カ私見ヲ加ヘシモ,シカモ猶ホ明確ナル區別ヲ認ムルヲ得ザルノ嘆アリ.

其ノ問題トハ次ノ如シ.

"今有方內容等三圓. 只云方面一尺. 問圓徑"(1).



即チ,正方形內ニ,其ノ一對角線上ニ中心ヲ有スル三ツノ等圓ヲ圖ノ如ク畫キ,其正方形ノ一邊ノ長サヲ一尺ナリトセルトキ各圓ノ直徑幾何ナルカトノ意ナリ.

"答曰. 圓徑四寸一分四厘二毫一絲三忽五六二三七三〇九五."

此答數ヲ得ルニ種々ノ術アリ.

（術名ナシ）

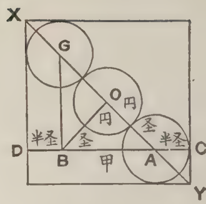
"第一 術曰. 置一寸爲圓徑率. 乘斜率爲甲率. 加圓徑率爲方面率. 爲法 置方面. 爲實. 實如法而一. 爲因法. 乘圓徑率. 得圓徑. 合問.

"解圖詳也."

求メラレタル圓徑ヲ假リニ一寸ナリトシ,斜率 ($=\sqrt{2}$ 正方形ノ一

(1) " " ヲ附セル部分ハ藤田ノ陳述ナリ, 但シ明ニ誤記ナリト思ヘル部分ハ多少卑見ヲ加ヘテ改訂セリ.

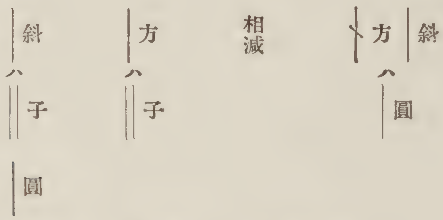
邊即チ所謂方面ヲ1トスレバ其對角線即チ所謂斜ハ$\sqrt{2}$ナレバナリ)ヲ圓徑1ニ乘ジ,之ヲ甲即チABトシ,之ニ圓徑1寸ヲ加ヘ,方面CDノ率(即チ割)トス,之ヲ除法ノ法トシ,方面一尺ヲ除法ノ實トシ,除法ノ結果ハ即チ所要ノ因數ナリ. 之ヲ圓徑率1寸ニ乘ジテ單位ノ名附セラルヽナリ. 簡約スレバ所要ノ圓徑ハ $\dfrac{1^{尺}}{1\times\sqrt{2}+1}$ ナリト云フナリ.

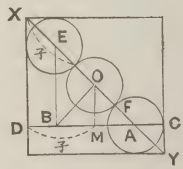


"又術"(術名ナシ.)

"第二 術曰. 置方面. 自乘之. 倍之. 開平方. 爲斜. 內減方面. 爲圓徑. 合問.

"解圖





"故本術. 斜內. 減方面. 爲圓徑也."

圖ニ於テ

$$CM=DM=XE=YF.$$

故ニ　　斜 $XY=2\cdot CM+$ 圓徑 $EF,$

然ルニ　　方 $CD=2\cdot CM,$

故ニ　　$XY-CD=$ 圓徑 $EF.$

此理ニ依リテ術文ハ CD ガ方即チ方面ナルヲ以テ,

$$斜=\sqrt{2\cdot(方面)^2},\quad 斜-方面=圓徑$$

ナリトイフナリ.

"又術. 天元

"第三 術曰. 立天元一. 爲圓徑. 加方. 爲斜. 寄左. 列方面. 乘斜率. 爲斜. 與寄左相消. 得歸除式. 上實如下法而一. 爲圓徑. 合問.

"解圖同前條."

天元一トハ我等ガ現時用ユル所ノ未知數ヲ代表スル x ノ如シ. サレバ, x ヲ圓徑トシ, 方 a (本題ニ於テハ 1尺) ヲ加フレバ對角線ナリ. 又他面ニ於テ, 方 a ニ斜率 $\sqrt{2}$ ヲ乘ズレバ同ジク對角線ナリ. 此兩者ヲ相消シ即チ相等シト置ケバ一元一次方程式ヲ得, 除法ヲ施コシ圓徑 x ヲ得ト云フコトナリ. 故ニ本術ハ方程式

$$x+a=a\sqrt{2}$$

ヲ解クコトヲ示セリ.

"又術. 天元.

"第四 術曰. 立天元一. 爲圓徑. 以減方. 爲甲. 自乘之. 爲圓徑冪二段. 寄左。列圓徑. 自之. 倍之. 與寄左相消. 得開方式. 平方開之. 得圓徑. 合問.

"解圖同第一圖."

本術モ亦天元術ニシテ術文ニ從ヒテ次ノ開方式即チ現時所謂二次方程式ヲ得. x ヲ圓徑トシ a ヲ方トスレバ

$$(a-x)^2$$

ハ圓徑冪二段ニシテ, 圓徑ヲ自乘シテ二倍セルモノハ

$$2x^2$$

ナリ. 故ニ

$$(a-x)^2=2x^2.$$

之ヲ平方ニ開キテ x ヲ得.

"又術. 天元.

"第五 術曰. 立天元一. 爲圓徑. 加方. 爲斜. 列圓徑. 以減方面. 爲斜率因圓徑. 乘斜. 爲方面因圓徑二段. 寄左。列圓徑. 乘方面. 倍之. 與寄左相消. 得開方式. 平方開之. 得圓徑. 合問.

"解圖同第一圖."

x ヲ圓徑トシ a ヲ方トスレバ

$$x+a=\text{斜},$$

$$a-x=\sqrt{2}\,x,$$

然ラバ

$$\sqrt{2}\,x\,(a+x)=2\,a\,x$$

ナリト. コハ第一術ノ圖ニ於テ

$$AB\,/\,AG=CD\,/\,XY,$$

即 $$AB\cdot XY=CD\cdot AG,$$

即 $$(a-x)\,(x+a)=a\cdot 2x$$

ヨリ得タルモノナルベシ.

"又術. 天元.

"第六 術曰. 立天元一. 爲圓徑. 以減方. 爲斜率因圓徑. 乘方. 爲圓徑因斜. 寄左。 列圓徑. 加方面. 爲斜. 乘圓徑. 爲圓徑因斜. 與寄左相消. 得開方式. 平方開之. 得圓徑. 合問.

"解圖同第一圖."

前ノ如クシテ順次

$$a-x=\sqrt{2}\,x\,,$$

$$\sqrt{2}\,x\cdot a=x\cdot \text{斜}\,.$$

$$x+a=\text{斜}$$

ヲ得. 故ニ

$$\sqrt{2}\,x\cdot a=x\,(x+a).$$

"又術. 天元求積.

"第七 術曰. 立天元一. 爲圓徑. 加方. 爲斜. 乘圓徑. 以減方冪. 爲黒積. 寄左。 列方. 乘圓徑. 爲黒積. 與寄左相消. 得開方式. 平方開之. 得圓徑. 合問.

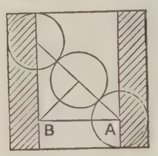
斜圓
八
方斜率圓
變
甲方
變
中貫

以減方冪

方冪
斜圓
八
黒積
寄左

圓方
八
黒積
相消數
"

解圖



圓, 斜, 方, 斜率, 甲, 中責, 及ビ黑積トアルハ夫々圓ノ直徑, 對角線, 正方形ノ一邊, $\sqrt{2}$, 線分 AB, 圖中央部ノ面積, 及ビ黑色ニ塗レル部分ノ面積ナリ. 前ノ如クシテ順次

$$斜\cdot x=(a\sqrt{2})\cdot x=a(\sqrt{2}x)=AB\cdot a=中責,$$

$$a^2-斜\cdot x=黑積$$

又

$$x\cdot a=黑積$$

故ニ二次方程式

$$a^2-(x+a)x=ax$$

ヲ得.

“又術. 天元維乘.

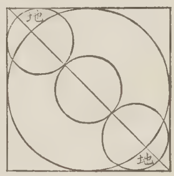
“第八 術曰. 立天元一. 爲圓徑. 加方. 爲斜. 內減圓徑三段. 餘爲天二段. 乘方. 爲圓徑冪. 寄左。列圓徑. 自之. 與寄左相消. 得開方式. 平方開之. 得圓徑. 合之.

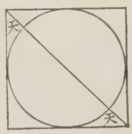
大圓ノ中ニ文句アリ「此圓徑者與方等故地二段者方與斜差也. 依此二段者圓徑也」

方	圓
地	天

故 方天 ハ 地圓 倍之

方天 ハ 地圓 變 圓巾 也





故本術. 斜內. 減圓三段. 爲天二段. 乘方面. 爲圓徑冪也.”

先ヅ天圓中ノ文句ニヨリテ

$$大圓徑=方,$$

$$2\times地=斜-方=x.$$

又比例式

$$\frac{x}{a}=\frac{天}{地}$$

ニヨリテ

$$a \times 天 = 地 \times x.$$

故ニ

$$2 \times a \times 天 = 2 \times 地 \times x = x \times x = x^2.$$

故ニ

$$(x+a) - 3x = 天ノ二倍,$$

從ツテ

$$\{(x+a) - 3x\}\, a = x^2$$

ヲ得.

"又術. 天元.

"第九 立天元一. 爲圓徑. 加方. 爲斜. 加又方. 爲下矢. 乘圓徑. 爲方冪. 寄左. 列方. 自之. 與寄左相消. 得開方式. 平方開之. 得圓徑. 合問.

解義

方巾也

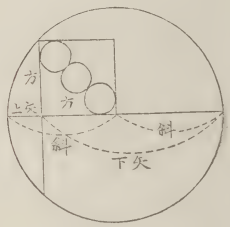
故下矢ニ乘圓徑(乃上矢)爲

上矢下矢ハ方巾也.

方斜ハ下矢

故圓徑ヲ上矢ト見

斜方ハ上矢變圓徑



"

解義ニ從ヘバ

$$斜 - a = 上矢 = x,$$

$$a + 斜 = 下矢,$$

$$下矢 \times 上矢 = a^2.$$

故ニ

$$下矢 \times x = a^2,$$

故ニ

$$\{(x+a) + a\}x = a^2$$

ヲ得.

"又術. 點竄.

"第十 置圓徑、乘斜率、加圓徑二段、爲方斜、 丨圓斜率 ‖圓 寄左。

列方、乘斜率、爲方斜、 丨方斜率

與寄左相消、得 丨圓斜率 ‖圓 $-$丨方斜率

變之 丨圓斜率 丨圓斜率巾 $-$丨方斜率

偏省斜率 丨圓 丨圓斜率 $-$丨方

分之、丨圓 $-$丨方 左 丨斜率圓 右

左自之 丨圓巾 $-$‖方圓 丨方巾 爲生

右自之 ‖圓巾 爲尅

生尅相消 $-$丨方巾 ‖圓方 丨圓巾

(本術) 置方. 自之. 爲負實. 以方二段. 爲正法. 以一算. 爲正廉. 平方開之. 得圓徑. 合問."

其意味ハ次ノ如シ

$$x\sqrt{2}+2x=斜,$$

$$a\sqrt{2}=斜.$$

故ニ

$$x\sqrt{2}+2x-a\sqrt{2}=0.$$

故ニ

$$x\sqrt{2}+x(\sqrt{2})^2-a\sqrt{2}=0,$$

故ニ

$$x+x\sqrt{2}-a=0,$$

故ニ

$$x-a=-x\sqrt{2},$$

故ニ

$$x^2-2ax+a^2=2x^2,$$

故ニ

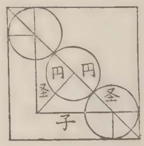
$$-a^2+2ax+x^2=0,$$

故ニ $-a^2$ ヲ絕對項(即チ實), $2a$ ヲ x ノ係數(即チ法), 1 ヲ x^2 ノ係數(即チ廉)トナシ, 此二次方程式ヲ解キ x ヲ得. 文中生尅トアルハ猶ホ正負ト云フガ如シ.

サレバ天元術トイフモ點竄術ト云フモ共ニ現時ノ代數學中ノ方程式解法ナルガ, 其ノ術文ニ於テ前者ハ後者ヨリモ自由ヲ缺キ, 式ヲ立ツル每ニ意味ヲ附スベキモ, 後者ハ方程式ノ移項, 兩邊ノ自乘等ニ於テ頗ル自由ナルモノアリテ實法廉等ノ諸級ノ係數ヲ述ブルニ止マル. 又特ニ天元術ノ術文ニ於テハ「立天元一」ノ語ガ通例頗ル目立ツモノナリ. 此等ニ比較シテ先キニ掲ゲタル第一及第二ノ術名ナキ解法ハ算術的解法トモイフヲ得ン.

"又術. 兩式.

"第十一 (虛術)立天元一、爲子 〇 |



自之、爲子冪 〇〇 |、寄左。

列圓徑、自之、倍之、爲子冪 圓巾 ||、

與寄左相消、得 圓巾 || 〇 |、前式。

列方、內減子、爲圓徑 方 | 丨、

與圓徑相消 方 | 圓 丨 丨 後式。

前式後式相併 圓巾 || 方 | 圓 丨 定前式。

後式下級、與定前式上級相乘、得 圓方 || 圓巾 | 爲生數。

後式上級、乘定前式下級、得 圓巾 || 爲尅數。

生尅相消 方巾 | 圓方 || 圓巾 | 正一位消 負二位寄

“(本術) 立天元一、爲圓徑〇 ―|、加方二段、得 ‖(方) ―|、乘圓徑、得〇 ‖(方) ―― 寄左。列方、自之、得 |(方巾) 與寄左相消、得開方式 ⊥(方巾) ‖(方) ―|、平方開之、得圓徑、合問。”

案ズルニ兩式術トハ前式, 後式ト稱スル二箇聯立方程式ヲ立テ, 之レニ我等ガ現時加減法ト稱スル消去法ヲ施シテ簡單ナル方程式ヲ求メ, 終ニ之ヲ解ク一種ノ點竄術ナリト見ルヲ得ベシ.

先ヅ虛術ト題シ本術ニ到達スルノ路ヲ明示ス.

圖中ノ子ヲ $0+y$ トス. 之ヲ自乘シテ子羃 $0+0\,y+y^2$ トス[1]. 又圓徑 x, y ヲ自乘シ, 二倍スレバ $2\,x^2$ ナルガ之亦子羃ナリ. 故ニ方程式

$$-2\,x^2+0\cdot y+y^2=0 \qquad \text{(前式)}$$

ヲ得. 之ヲ前式トス.

方 a ヨリ子ヲ減ズレバ $a-y$ ナルガコハ圓徑ナリ. 然ルニ圓徑ハ x ナリ. 故ニ方程式

$$(a-x)-y=0 \qquad \text{(後式)}$$

ヲ得. 之ヲ後式トス.

前式ト後式ト(ノ絕對項)ヲ相加フレバ

$$-2\,x^2+(a-x). \qquad \text{(定前式)}$$

之レヲ定前式ト云フ.

後式ノ下級即チ絕對項 $a-x$ ト定前式ノ上級即チ $a-x$ トヲ乘ズレバ $x^2-2\,a\,x+a^2$ ナリ. 之ヲ生數(即チ稍々當ラザルモ正數トイフヲ得ン)トス.

次ギニ後式ノ上級即チ y ノ係數 -1 ($-y$ ヲ取ラズ) ニ定前式ノ下級即チ絕對項 $-2\,x^2$ ヲ乘ズレバ $2\,x^2$ ナリ. 之ヲ尅數(負數トイフヲ得ン)トス.

生尅兩數ヲシテ相消セシメ,

$$-x^2-2\,a\,x+a^2=0.$$

故ニ次ノ本術ヲ立ツ, コハ全ク此ノ最後ノ方程式ヲ作ル方法ヲ意味ヲ附セズシテ陳述スルナリ.

---

(1) 斯 0 ヲ附スルハ和算ノ特色ナリ.

先ヅ圓徑ヲ $0+a$ トシ方二段 $2a$ ヲ加ヘ $2a+x$ トス圓徑 $0+x$ ヲ乘ジ $0+2ax+x^2$ トス. 又別ニ方冪 a^2 ヲ得, 兩者相等シト置キ方程式

$$-a^2+2ax+x^2=0$$

ヲ得. (x ト $=0$ トハ書キ表ハサズ.) 之ヲ解キテ x ヲ得トイフナリ.

定前式ノ意義及ビ其ノ使用ノ理由上記ノ儘ニテ全ク不明ナリ. 之ヲ有理ニ解釋センニハ, 先ヅ一旦得タル後式ノ各項ノ位ヲ昇シ (即チ $0+y$ ヲ乘ジ)テ (1)

$$0+(a-x)y-y^2=0$$

トシ, 然ル後之ト先キニ得置キタル前式

$$-2x^2+0\cdot y+y^2=0$$

トヲ邊々相加フレバ

$$-2x^2+(a-x)y=0$$

(之ヲ定前式トイフモノト見ルベシ). 斯クシテ此式ト後式

$$(a-x)-y=0$$

トノ間ニ加減法ニヨリテ y ヲ消去スレバ

$$(a-x)^2-(-2x^2)(-1)=0$$

即チ

$$-x^2-2ax+a^2=0$$

ヲ得ルナリ.

"又術. 括術.

"第十二 術曰. 置方. 乘定法〇箇 四一 四二. 得圓徑. 合問 此術四位 合眞數."

括術トハ既ニ他種ノ術ニテ得タル結果ヲ簡約シテ定法ナルモノヲ求メ, 成ルベク直接ニ既知數ヨリ未知數ヲ得ルノ術ナリ. 前諸術ニヨリテ本問題ニ就テハ

$$x=a(\sqrt{2}-1)$$

故ニ

$$\sqrt{2}=1.4142$$

---

(1) 蓋シ和算家ガ使用スル算盤上ニテ位ヲ昇スコトハ餘リ簡單ナルコトナリシヲ以テ特ニ之レヲ表ハサザリシモノト見ユ. 兩式ノ語ニ就キテハ既ニ東京數學物理學會記事第 2 期第 5 卷 (1910) 頁 257 及頁 263 ニ於テ説明シタルコトアリ.

トシ

$$0{\cdot}4142$$

ヲ定法ト名ヅケ,之ヲ既知數タル正方形ノ一邊ニ乘ズレバ所要ノ圓徑ヲ得,小數點以下第四位マデ眞數ニ合ストイフナリ.

"又術. 零約.

"第十三 術曰. 置方面. 乘二十九. 得數如七十〇而一. 得圓徑. 合問 此術四位合眞數."

零約術ニ關シテハ余既ニ東北數學雜誌第六卷 (1914–15) 頁 188–231 並ニ同第七卷 (1915) 頁 1–17 ニ於テ詳論シタリ, 至ツテ連分數展開ト密接ノ關係ヲ有ス, 其ノ第 209 頁ニ於テ見ルガ如ク $\sqrt{2}$ ニ對スル數列トシテ

$$\frac{1}{1},\ \frac{3}{2},\ \frac{4}{3},\ \frac{7}{5},\ \frac{10}{7},\ \frac{17}{12},\ \frac{24}{17},\ \frac{29}{41},\ \frac{58}{41},\ \frac{99}{70},\ \cdots\cdots$$

ヲ得,即チ第十位ノモノ

$$\frac{99}{70}$$

ヲ取リテ $\sqrt{2}$ ノ近似値(此分數ハ所謂多率ナルヲ以テ過剩近似値ナリ)トスレバ

$$\sqrt{2}-1=\frac{29}{70},$$

故ニ本問題ハ方面 a ニ 29 ヲ乘ジ 70 ニテ除スレバ圓徑ヲ得ベシトイフナリ.

括術ニ於テモ,將タ又零約術ニ於テモ其術文ニ到達スルノ前ニ於テ,前陳ノ天元術,點竄術等ヲ用ヒ $\sqrt{2}-1$ ヲ得タル後之ヲ變ズルモノナリト見ユ.

"又術. 統術.

"第十四 術曰. 置一個. 爲圓徑. 以減方. 餘爲子 九. 自之. 爲圓徑冪二段 八十一. 寄左。 與圓徑冪二段相消 七十九. 爲甲數."

圓徑ヲ 1 (與ヘラレタル方面ガ 1 尺ナレバ一單位下ダシテ 1 寸ノツモリ)トシ之レヲ方面 1 尺ヨリ減ジテ 9 ヲ得,自乘スレバ 81 ハ圓徑

x ノ自乘ノ 2 倍ニ當ル. 此ノ數ヲ甲數トス

$$(10-1)^2-2\times 1^2=79.$$

"置二個. 爲圓徑. 以減方. 餘爲丑八. 自之. 爲圓徑冪二段六十四. 寄左。與圓徑冪二段相消五十六. 爲乙數."

圓徑ヲ 2 (2 寸ノツモリ)トシ, 前ノ如クシ

$$(10-2)^2-2\times 2^2=56,$$

之ヲ乙數トス.

"置三個. 爲圓徑. 以減方. 餘爲寅七. 自之. 爲圓徑冪二段四十九. 寄左。與圓徑冪二段相消三十二. 爲丙數."

圓徑ヲ 3 (3 寸ノツモリ)トシ, 前ノ如クシ

$$(10-3)^2-2\times 3^2=31.$$

"甲數內減乙數. 餘二十三爲左. 乙數內減丙數. 餘二十五爲右. 左內減右. 餘二負爲廉數. 右三段內減左五段. 餘四十〇負爲法數. 置甲數倍之. 內併減法廉二百〇〇正. 爲實數. 依遍約術. 得實一百〇〇正. 法二十〇負. 廉一負. 平方開之. 得圓徑. 合問."

$$(79-56)-(56-31)=-2,$$
$$3(56-31)-5(79-56)=-40,$$
$$2\times 79-(-2)-(-40)=+200.$$

是ニ於テ $+200$, -40, -2 ノ公約數ヲ去リ 100, -20, -1 トシ, 之レヲ夫々二次方程式ノ係數トシ

$$100-20x-x^2=0$$

ヲ得. 之レヲ解ケバ圓徑ヲ得ト云フナリ.

遍約術トハ二ツ以上ノ整數ノ公約數ヲ去ル術ヲイフ.

上記ノ統術解ハ彼ノ儘ニテハ殆其ノ正否ヲ判別スル能ハズ. 余ハ未ダ嘗テ統術ナルモノヲ說明セシコトナク, 又之ガ說明ハ稍々長文ニ涉リ此處ニ挿入スルニハ不適當ナルヲ以テ, 附錄トシテ最後ニ委敷之ガ說明ヲ試ミントス.

"又術. 綴術.

"第十五 術曰. 置九十九. 自之. 倍之. 爲除法.

置方面. 爲甲數。
置甲數. 九十九乘. 七十除. 而爲乙數。
置乙數. 以除法除之. 一乘一除. 而爲一差。
置一差. 以除法除之. 一乘二除. 而爲二差。
置二差. 以除法除之. 三乘三除. 而爲三差。
置三差. 以除法除之. 五乘四除. 而爲四差。
置四差. 以除法除之. 七乘五除. 而爲五差。
所求. 置乙數. 内累減甲數及諸差. 餘爲圓徑. 合問.

"數解

方／ 甲數一尺

甲數九九／七〇 乙數一尺四寸一四二八五七一四二八五七一四二

乙數一／除法一 一差〇寸〇〇〇七二一五〇〇七二一五〇〇

一差一／除法二 二差〇寸〇〇〇〇〇〇〇一八四〇三七五二

二差三／除法三 三差〇寸〇〇〇〇〇〇〇〇〇〇〇〇九三八

所求. 置乙數. 内累減甲數及諸差. 得數

四寸一四二一三五六二三七〇九五餘

爲圓徑. 合問.

"右術. 三差而眞數ニ合フコト凡一十七位."

綴術トハ余ガ屢々繰リ返ヘシテ述ベタルガ如ク ([1]), 無限級數ニ展開スルコトヲ用ユル術ナリ. 但シ無限級數トイフハ一定ノ法則ニ遵ヒテ誘導シタル數列ノ總和ヲ求ムルモノトハカギラズ. 總和ヲ求メズシ

([1]) タトヘバ東京數學物理學會記事第 2 期第 4 卷 (1908) 頁 446–453, 第 5 卷 (1909) 頁 43–57, 同卷 (1910) 頁 407–414, 第 6 卷(1911)頁 144—152 等ヲ參照スベシ.

テ，此ノ如キ數列ノ某番目ノ一項(之ヲ以テ極限値ト見做シ)ヲ以テ所要ノ數トスルコトモアレドモ，多クハ其ノ數列ノ總和（タトヘ極限値マデ求メタリトハセザルモ，收斂ヲ假定シ初メ數項ノ和ニテ眞數若干位ニ到達セルモノトス)ヲ求ムルモノナリ．本題ニ對スル上記ノ解法亦其ノ一例ナリ．

今方面ヲ a トシ圓徑ヲ x トスレバ前記ノ術文ハ次ノ如クナル．

2×99^2…………除法（後ニ之レヲ除スル定法又ハ定率トスルノ意）

a………………甲數

$a\times\dfrac{99}{70}$………乙數

$\dfrac{乙數}{除法}\times\dfrac{1}{1}$……一差

$\dfrac{一差}{除法}\times\dfrac{1}{2}$……二差

$\dfrac{二差}{除法}\times\dfrac{3}{3}$……三差

$\dfrac{三差}{除法}\times\dfrac{5}{4}$……四差

$\dfrac{四差}{除法}\times\dfrac{7}{5}$……五差

然ラバ

$$x=(乙數)-(甲數)-(一差)-(二差)-(三差)-(四差)-(五差)$$

ナリトイフナリ．故ニ

$$2\times 99^2=s,\quad \frac{99}{70}=t$$

トシテ代入スレバ

$$x=at-a-\frac{at}{s}\times\frac{1}{1}-\frac{at}{s^2}\times\frac{1.1}{1.2}-\frac{at}{s^3}\times\frac{1.1.3}{1.2.3}$$
$$-\frac{at}{s^4}\times\frac{1.1.3.5}{1.2.3.4}-\frac{at}{s^5}\times\frac{1.1.3.5.7}{1.2.3.4.5}$$

トナル．故ニ

$$x=a\Big\{t\Big(1-\frac{1}{1}\cdot\frac{1}{s}-\frac{1.1}{1.2}\cdot\frac{1}{s^2}-\frac{1.1.3}{1.2.3}\cdot\frac{1}{s^3}$$

$$-\frac{1.1.3.5}{1.2.3.4}\cdot\frac{1}{s^4}-\frac{1.1.3.5.7}{1.2.3.4.5}\cdot\frac{1}{s^5}\Big)-1\Big\},$$

然ルニ前諸術ニヨリテ

$$x=a(\sqrt{2}-1)$$

ナリ．故ニ

$$\sqrt{2}=t\Big(1-\frac{1}{1}\cdot\frac{1}{s}-\frac{1.1}{1.2}\cdot\frac{1}{s^2}-\frac{1.1.3}{1.2.3}\cdot\frac{1}{s^3}$$
$$-\frac{1.1.3.5}{1.2.3.4}\cdot\frac{1}{s^4}-\frac{1.1.3.5.7}{1.2.3.4.5}\cdot\frac{1}{s^5}\Big)$$

トセシナリ．

偖テ

$$t=\frac{99}{70}$$

ハ先キニ零約術ニヨル解ニ於テ用ヒタル$\sqrt{2}$ノ多率ナリ．故ニ之ヲ標準トシ$\sqrt{2}$ノ近似値ヲ得ルニハ(1)

$$\sqrt{2}=\sqrt{t^2-(t^2-2)}=t\Big\{1-\frac{t^2-2}{t^2}\Big\}^{\frac{1}{2}}$$
$$=t\Big\{1-\frac{1}{2}\cdot\frac{1}{\sigma}-\frac{1.2}{2.4}\cdot\frac{1}{\sigma^2}-\frac{1.1.3}{2.4.6}\cdot\frac{1}{\sigma^3}$$
$$-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{\sigma^4}-\frac{1.1.3.5.7}{2.4.6.8.10}\cdot\frac{1}{\sigma^5}\Big\}$$
$$=t\Big\{1-\frac{1}{1}\cdot\frac{1}{2\sigma}-\frac{1.1}{1.2}\,\frac{1}{(2\sigma)^2}-\frac{1.1.3}{1.2.3}\cdot\frac{1}{(2\sigma)^3}$$
$$-\frac{1.1.3.5}{1.2.3.4}\,\frac{1}{(2\sigma)^4}-\frac{1.1.3.5.7}{1.2.3.4.5}\cdot\frac{1}{(2\sigma)^5}\Big\},$$

但シ

$$\sigma=\frac{t^2}{t^2-2}.$$

之レヲ前記ノ藤田ノ方法ト對照スルニ相合ハザルヲ見ルベシ，藤田ガ如何ニシテ上ノ方法ヲ得シカ不明ナリ．

何トナレバ對照ニヨリテ

(1) 尙ホ此ノ方法ニツキテハ東北數學雜誌第 11 卷 (1917) 頁 17-37 ニ於ケル說述「安島萬藏及ビ松永貞之允」[頁24] ヲ參照スベシ．

$$s=2\sigma=\frac{2t^2}{t^2-2}$$

即チ
$$2\times 99^2=\frac{2\times 99^2}{99^2-2\times 70^2}$$

トセザルベカラザルヲ以テナリ. 所謂除法ナルモノヲ變更スルノ要アルガ如シ. 恐ラクハ術文ノ誤記ナルベシ.

余未ダ嘗テ統術ナルモノヲ説明セシコトアラズ, 今此機會ヲ利用シテ稍々委敷其説明ヲ試ミントス, 典據スルトコロノモノハ絳老餘算統術總括ト題スル寫本ニシテ寛保元年 (1741) 重光作噩畢皐吉トアリ. 關流正統三傳松永良弼ノ編次ニシテ同正統四傳山路主住ノ全校ナリ.

統術ニ屬スル條項ヲ, 歸除, 平方, 立方等ニ區別シ各々常式及ビ變格アリ. 例解ヲ先ニキシ法則ヲ後ニシテ以下逐次説明ヲ試ミン.

例. 雞兎共ニ百アリ, 足數併テ二百七十二ナリ, 雞兎各幾羽ナルカ.

答. 雞六十四.

歸除常式

兎ヲ 1 トセバ以テ 100 ヲ減ジテ雞 99 ナリ.

兎足數 1×4, 雞足數 99×2, 相併セテ 202 ナリ.

所題數 272 ヨリ少キコト 70 ナリ. 之ヲ甲トス.

兎ヲ 2 トセバ以テ 100 ヲ減ジテ雞 98 ナリ.

兎足數 2×4, 雞足數 98×2, 相併セテ 204 ナリ.

所題數 272 ヨリ少ナキコト 68 ナリ, 之ヲ乙トス.

甲 70 ヲ 2 倍シテ, 内乙 68 ヲ減ジテ餘 72, 實トス.

乙ノ内甲ヲ減ジテ負 2, 法トス

實ヲ法ニテ割リテ $72\div 2$ 即 36, 兎ノ數トス.

之ヲ一般ニイハバ二元聯立方程式

$$ax+by=c$$
$$a'x+b'y=c'$$

ヲ解クニ當リ, 順次 $x=1, 2$ ナル値ヲ與ヘ, 之ニ對應スル y ノ値

$$\frac{c-a}{b},\quad \frac{c-2a}{b},$$

ヲ求メ, 之レ等ヲ $a'x+b'y$ ニ代入スレバ

$$a'+\frac{b'}{b}(c-a), \quad 2a'+\frac{b'}{b}(c-2a),$$

之レヲ c' ヨリ減ジテ各殘リヲ ξ, η トスレバ

$$\xi=(c'-a')-\frac{b'}{b}(c-a), \quad \eta=(c'-2a')-\frac{b'}{b}(c-2a).$$

然ラバ

$$-x=\frac{2\xi-\eta}{\eta-\xi}$$

ナリトセルナリ, 實際

$$\frac{2\xi-\eta}{\eta-\xi}=\frac{bc'-cb'}{ab'-a'b}$$

トナル.

斯ク最後ニ一ツノ割算ヲ用ヒテ未知數ヲ得ルガ故ニ歸除(又ハ飯除)ト云ヒ,最モ普通ノ方法ナルガ故ニ常式トイフ.

次ギニ同例題ヲ亦歸除ニ屬スル一ツノ變格(即チ變態)ニテ解ク方法ヲ述ベン.

歸除變格第一

兎ヲ 3 トセバ 100 ヨリ減ジテ雞 97 ナリ.

足數併セテ 206, 所題數 272 ヨリ少キコト 66, 甲トス.

兎ヲ 6 トセバ 100 ヨリ減ジテ雞 94 ナリ.

足數併セテ 212, 所題數 272 ヨリ少キコト 60, 乙トス.

甲 66 ヲ 2 倍シテ, 內乙 60 ヲ減ジ, 餘 72 ニ 3 (初メ兎ノ數トセシ數)ヲ乘ジテ 216, 實トス.

乙ノ內甲ヲ減ジテ負 6, 法トス.

實ヲ法ニテ割リ 216÷6 即 36, 兎ノ數トス.

之レヲ一般ニイハバ, 前記ノ聯立方程式ヲ解クニ當リテ, 順次 $x=r$, $2r$ ナル値ヲ與ヘ, 之ニ對應スル y ノ値

$$\frac{c-ra}{b}, \quad \frac{c-2ra}{b}$$

ヲ求メ, 之レ等ヲ $a'x+b'y$ ニ代入スレバ

$$ra'+\frac{b'}{b}(c-ra), \quad 2ra'+\frac{b'}{b}(c-2ra),$$

之レ等ヲ c' ヨリ減ジテ各殘リヲ ξ,η トスレバ

$$\xi=(c'-ra')-\frac{b'}{b}(c-ra),\qquad \eta=(c'-2ra')-\frac{b'}{b}(c-2ra).$$

然ラバ

$$-x=\frac{(2\xi-\eta)r}{\eta-\xi}$$

ナリトセルナリ. 實際

$$\frac{(2\xi-\eta)r}{\eta-\xi}=\frac{bc'-cb'}{ab'-a'b}$$

トナル.

故ニ此ノ變格ハ r ナル任意ニ定メ得ル因數ヲ採用シタルモノナリ.

歸除變格第二

兎ヲ 7 トセバ雞 93, 其ノ足數 214 ニシテ所題數 272 ヨリ少ナキコト 58, 甲トス.

兎ヲ 8 トセバ雞 92, 其ノ足數 216 ニシテ所題數 272 ヨリ少ナキコト 56, 乙トス.

甲ヲ 2 倍シ, 餘乙ヲ減ジ 60, 實トス

乙ノ内甲ヲ減ジ 2, 法トス.

實ヲ法ニテ割リ 30, 之ニ 7 (初メ兎ノ數トセシ數)ヲ加ヘ 1 (定數)ヲ減ズレバ兎 36 ヲ得.

コレヲ一般ニイハバ, 前記ノ聯立方程式ニ於テ, 順次 $x=\alpha,\ \alpha+1$ ナル値ヲ値ヘ, 之ニ對應スル y ノ値

$$\frac{c-\alpha a}{b},\qquad \frac{c-(\alpha+1)a}{b}$$

ヲ求メ; 之レ等ヲ $a'x+b'y$ ニ代入スレバ

$$\alpha a+\frac{b'}{b}(c-\alpha a),\quad (\alpha+1)a+\frac{b'}{b}(c-(\alpha+1)a),$$

之レ等ヲ c' ヨリ減ジテ各殘リヲ ξ,η トスレバ

$$\xi=(c'-\alpha a)-\frac{b'}{b}(c-\alpha a),\qquad \eta=(c'-(\alpha+1)a)-\frac{b'}{b}(c-(\alpha+1)a).$$

然ラバ

$$x=-\frac{2\xi-\eta}{\eta-\xi}+\alpha-1$$

ナリトイフナリ. 實際

$$-\frac{2\xi-\eta}{\eta-\xi}+\alpha-1=\frac{cb'-bc'}{ab'-a'b}$$

ナリ. α ハ任意ノ數ナリ.

故ニ歸除通術トシテハ

$$2\xi-\eta, \qquad -\xi+\eta$$

ヲ揭グ.

次ギニハ二次方程式ヲ解クコトヲ要スル問題ニ對スルモノヲ述ベン.

平方常式

順次 $x=1,2,3$ ナル値ヲ與ヘ既知數ト未知數トノ間ニ成立スベキ關係(必ズシモ一次方程式ニアラズ)ノ一ツヨリ之ニ對スル y ノ値, y, y', y'' ヲ求メ之レヲ第二ノ關係ニ代入シテ絕對項ニ起コル差(較トイフ) ξ, η, ζ (之ヲ各甲乙丙數トイフ)ヲ求メ, 更ニ

$$u=\xi-\eta, \qquad v=\eta-\zeta$$

(之レヲ各左右數トイフ)ヲ求メ, 進ンデ

$$p=u-v, \quad q=3v-5u, \quad r=2\xi-p-q$$

ヲ求ムレバ, 所要ノ x ハ二次方程式

$$r+qx+px^2=0$$

ノ根ナリ.

例. 直(矩形)アリ其面積 63 寸(平方寸)ニシテ長平(二邊)ノ差 2 寸ナリ, 長平各幾寸ナルカ. 答. 平 7 寸, 長 9 寸.

平長ヲ夫々 x, y トスレバ既知數ト未知數トノ間ニ成立スル關係ハ

$$y-x=2,$$
$$xy=63.$$

故ニ平 x ヲ假定スレバ之ニ 2 ヲ加ヘ y ヲ得, 之ヲ x ニ乘ジ, 其積ト 63 トノ差(即チ所謂較)ヲ ξ, η, ζ トス. (コハ面積ノ差ニシテ眺積トイヘリ.)

今本常式ニヨリテ $x=1,2,3$ ナル値ヲ與フレバ

$$\xi=60, \qquad \eta=55, \qquad \zeta=48,$$
$$u=5, \qquad v=7,$$
$$p=-2, \qquad q=-4, \qquad r=126$$

ヲ得テ二次方程式

$$126-4x-2x^2=0$$

ニ達ス.

一般ニ聯立方程式

$$\begin{cases} y=f(x) \\ \varphi(x,y)=a \end{cases}$$

ヲ取リ上ノ如クセバ

$$\xi=a-\varphi(1,f(1)),\quad \eta=a-\varphi(2,f(2)),\quad \zeta=a-\varphi(3,f(3)),$$

$$u=-\varphi(1,f(1))+\varphi(2,f(2)),$$

$$v=-\varphi(2,f(2))+\varphi(3,f(3)),$$

$$p=u-v,\quad p=3v-5u,\quad r=2\xi=p-q$$

ニシテ遂ニ二次方程式

$$r+qx+px^2=0$$

ヲ得タルナリ. 故ニ此ノ二次方程式ガ

$$\varphi(x,f(x))=a$$

ト等價ナルコトヲ知リ居タルモノナリ, ソノ等價ナルコトノ證明ハ次ノ如シ.

今

$$\varphi_x\equiv\varphi(x,f(x))\equiv Ax^2+Bx+C$$

ト置キ x ノ代ハリニ 1, 2, 3 ト置ケバ

$$\varphi_1=A+B+C,$$

$$\varphi_2=4A+2B+C,$$

$$\varphi_3=9A+3B+C$$

ナルヲ以テ, 之ヨリ

$$-2A=-\varphi_1+2\varphi_2-\varphi_3=p,$$

$$-2B=5\varphi_1-8\varphi_2+3\varphi_3=q,$$

$$-2C=-6\varphi_1+6\varphi_2-2\varphi_3=r-2a.$$

之ヲ

$$\varphi_x=a,$$

即チ

$$Ax^2+Bx+C=a$$

ニ代入スレバ

$$px^2+qx+r=0$$

トナル.

歸除常式モ亦此ノ方法ニヨリテ其ノ妥當ナルコトヲ示シ得ベシ.

平方變格第一

コハ常式ニ於ケル x ノ値 1, 2, 3 ノ代ハリニ $r, 2r, 3r$ ヲ置キ常式ノ時ノ如クシテ得ル所ノ二次方程式ヲ解キ, 其根ニ r ヲ乘ジテ所要ノ値トス.

平方變格第二

コハ常式ニ於ケル x ノ値 1, 2, 3 ノ代ハリニ $\alpha, \alpha+1, \alpha+2$ ヲ置キ常式ノ時ノ如クシテ得ル所ノ二次方程式ヲ解キ, 其根ニ α ヲ加ヘ 1 ヲ減ジテ所要ノ値トス.

故ニ**平方通術**トシテハ

$$\xi-2\eta+\zeta=-p,$$
$$-5\xi+8\eta-3\zeta=-q,$$
$$6\xi-8\eta+2\zeta=-r$$

ヲ記憶スベシ.

立方常式

ニ至リテハ上ト同様ニシテ, 聯立方程式

$$y=f(x)$$
$$\varphi(x,y)=a$$

ヲ得タリトスレバ方程式

$$\varphi(x,f(x))=a$$

ガ三次方程式トナル場合ニ用ヒラルヽ術路ヲ得タルモノナリ.

$$\varphi_x\equiv Ax^3+Bx^2+Cx+D=a$$

トシ

$$\varphi_1=A+B+C+D,$$
$$\varphi_2=8A+4B+2C+D,$$
$$\varphi_3=27A+9B+3C+D,$$
$$\varphi_4=64A+16B+4C+D$$

ヨリ A, B, C, D ヲ決定シテ之ヲ

$$Ax^3+Bx^2+Cx+D=a$$

ニ代入セル方程式ヲ解キタルナリ.

立方變格

第一及第二ニ於テハ常式ニ於ケル 1, 2, 3 ノ代ハリニ夫々 $r, 2r, 3r$ 及 $a, a+1, a+2$ ヲ置キテ全ク常式ノ場合ノ如ク進ミ三次方程式ヲ作リ之ヲ解キテ得ル所ノ根ニ r ヲ乘ズルカ又ハ $a-1$ ヲ加フルカニヨリテ所要ノ値ヲ得タルナリ.

故ニ立方通術トシテハ

$$\begin{aligned}-\xi+3\eta-3\zeta+\omega&=-p,\\ 8\xi-24\eta+21\zeta-6\omega&=-q,\\ -26\xi+57\eta-42\zeta-11\omega&=-r,\\ 24\xi-36\eta+24\zeta-6\omega&=-s\end{aligned}$$

ヲ記憶スベキナリ.

統術秘傳ニ**均差式**ト稱スルモノアリ, 亦歸除, 平方, 立方等ニ區分ス.

均差式歸除ニ於テハ未知數 x ノ二ツノ假定數ヲ夫々 x', x'' トシ前記ノ如ク ξ, η ヲ作リ

$$x=\frac{(2\xi-\eta)(x'-x'')}{\eta-\xi}+2x'-x''$$

ヲ以テ所要ノ答トス.

例. 雞兎共ニ 100 アリ, 足數 300 ナリ, 各頭數ヲ求ム.

答. 各 50.

何トナレバ雞ノ數ヲ x トシ兎ノ數ヲ y トスレバ

$$y=100-x$$

$$2x+4y=300.$$

故ニ $x'=8$ トスレバ足數眺 $\xi=84$

$x''=20$ トスレバ足數眺 $\eta=60.$

之レヨリ前公式ヲ適用シテ

$$x=50$$

ヲ得.

一般ニ聯立方程式

$$y=f(x)$$

$$\varphi(x,y)=a$$

トシ, 且ツ

$$\varphi_x \equiv \varphi(x, f(x)) = Ax + B$$

トスレバ

$$\varphi_x' = Ax' + B$$
$$\varphi_x'' = Ax'' + B.$$

之レヨリ

$$A = \frac{\varphi_x' - \varphi_x''}{x' - x''}, \quad B = \frac{x'\varphi_x'' - x''\varphi_x'}{x' - x''}$$

ヲ得. 故ニ

$$\frac{\varphi_x' - \varphi_x''}{x' - x''}x + \frac{x'\varphi_x'' - x''\varphi_x'}{x' - x''} = a$$

ヲ解クベキコトヽナル, 斯クスル代ハリニ前記ノ如ク ξ, η ヲ用ヒシモノナリ.

均差式平方ニ於テハ未知數 x ノ三ツノ假定數(之ヲ立標數トイフ)ヲ x', x'', x''' トシ $x'-x''=x''-x'''$ ナル關係ヲ保タシメ(此ノ故ニ均差トイフ), 前記ノ如ク ξ, η, ζ ヲ作リ從テ得ル所ノ二次方程式ヲ解キ之ニ均差 $x'-x''$ ヲ乘ジ其ノ積ニ $2x'-x''$ ヲ加フルモノナリ.

均差式立方等モ亦同樣ナリ.

均差式ニ至リテハ, 前陳普通ノ統術ヨリモ其ノ假定數遙カニ一般ナリ. 故ニ和算家ハ大ニ之ヲ秘トセシモナノナルベシ.

以上ハ松永良弼ノ編セル統術總括ニヨリテ記述セルモノナルガ, 藤田定資モ亦寛政壬子 (1792) 春正ニ於テ統術秘傳ナル書ヲ校正セリ, 原本何人ノ編著ナルカヲ知ラズ. 所解ノ問題七箇ニシテ總テ利息算ナリ. 今其内最モ簡單ナル第一題ヲ示サン. 但シ前述ノモノヨリ複雜ナリ.

假令金五兩貸シ利ヲ初年金三兩取ル. 殘金元利ニ利ヲ加ヘ (但シ金一步以下ハ利ヲ加ヘズ), 次年ニ金三兩取テ皆濟ナリ, 此ノ年何割ト問フ.

答曰. 年利一割四分二厘八毛五絲强 乃眞數七分之一.

所要ノ利率ヲ x トスレバ二次方程式

$$\{5(1+x)-3\}(1+x)-3=0$$

ヲ解クコトヽナル, 但シ二度目ニハ $\frac{1}{4}$ 兩(即チ一步)以下ニハ利ヲ加ヘズ. 故ニ $1+x$ ヲ乘ズルトキ注意スベシトイハル. 故ニ本題ニ於テハ

$$y=\{5(1+x)-3\}(1+x)-3.$$

先ヅ $$x_1'=\frac{1}{10} \text{ トスレバ } y_1'=-\frac{1}{4}\,(^1),$$

$$x_1''=\frac{2}{10} \text{ トスレバ } y_1''=\frac{3}{5}.$$

然ラバ $$\frac{y_1'x_1''-y_1''x_1'}{y_1'-y_1''}=\frac{11}{85}=0.129\cdots\cdots$$

是ニ於テ

$$x_2'=\frac{12}{100} \text{ トスレバ } y_2'=\cdots\cdots$$

$$x_2''=\frac{13}{100} \text{ トスレバ } y_3''=\cdots\cdots$$

然ラバ $$\frac{y_2'x_2''-y_2''x_2'}{y_2'-y_2''}=\cdots\cdots$$

是レ所要ノ解ナリトス.

第二題ハ次ノ如シ.

假令金五十兩ヲ貸シ利ヲ加ヘテ金六兩取リ, 殘元利ニ利ヲ加ヘ(但シ金一歩以下ハ利ヲ加ヘズ次年亦同ジ), 次年金三十兩取リ殘元利ニ利ヲ加ヘ三年金四十兩取皆濟ナリ. 年利何割ト問フ.

コハ所要ノ利率ヲ x トスレバ三次方程式

$$[\{50(1+x)-6\}(1+x)-30](1+x)-40=0$$

ヲ解クコトヽナル. 其ノ解法ハ亦前陳ト同様ナリ. 他ノ第三題ヨリ第七題マデ亦同様ナリ.

之ヲ以テ觀レバ本書ノ記述ハ二次以上ノ次數ヲ有スル方程式ノ根ノ略近値ヲ累次接近法ニヨリテ求ムル一種ノ方法ニシテ, 斯ノ如キヲモ統術トイヒシガ如シ.

此一種ノ累次接近法ニ於テハ y_1' ト y_2'' トガ其ノ符號ノ相反スルガ如ク x_1' ト x_2'' トヲ取ルベキモノニシテ, 三上義夫君ノ著 The Development of Mathematics in China and Japan (1913), pp. 166–167 及ビ Prof. D. E. Smith 及ビ三上氏ノ共著 A History of Japanese Mathematics (1914), pp. 169–170 ニ於テ中根彦循ニ歸セシメラル, ソハソガ享

(1) 本書ニハ $-\frac{3}{10}$ トセリ. 不明ナリ.

保 14 年 (1729) ニ著ハセル開方盈朒術中ニ含マルヽヲ以テナリ. 余ハ今同術ガ藤田定資ノ校正セル統術秘傳ニモ載セラレタルヲ記セルナリ. 但シ藤田ノ校正ハ中根ノ著ニ後ルヽコト約四十年ナリ.

此他ニ單ニ統術ト題スル書, 統術諺解, 統術演段, 統術招差等ノ書ヲ見ルモ, 統術トハ果シテ斯ノ如キ術ナリトハ斷定シ難シ, 藤田定資ガ著セルモノナリトスル算藪漫錄中ノ統術ト同人ノ著セル統術秘傳トニスラ上記ノ如キ相違アリ, 但余ハ前者ヲ以テ, 從ツテ松永ノ統術總括ヲ以テ統術ノ本態トナサントス.

大正七年三月

On Integral Inequalities between Two Systems of Orthogonal Functions,

by

KINNOSUKE OGURA, Ôsaka.

Let

$$f_1(x),\quad f_2(x),\quad \cdots\cdots,\quad f_n(x),\quad \cdots\cdots;$$

$$\varphi_1(x),\quad \varphi_2(x),\quad \cdots\cdots,\quad \varphi_n(x),\quad \cdots\cdots$$

be two different systems of orthogonal functions in the same interval $(a \leqq x \leqq b)$, such that

$$\int_a^b f_n^2(x)\,dx = \int_a^b \varphi_n^2(x)\,dx = 1, \quad (n = 1, 2, \cdots\cdots);$$

$$\int_a^b f_m(x) f_n(x)\,dx = \int_a^b \varphi_m(x)\,\varphi_n(x)\,dx = 0, \quad (m \neq n;\ m, n = 1, 2, \cdots\cdots).$$

If we put

$$f_n(x) = \varphi_n(x) + \psi_n(x),$$

then

$$\int_a^b f_n^2(x)\,dx = \int_a^b \varphi_n^2(x)\,dx + 2\int_a^b \varphi_n(x)\,\psi_n(x)\,dx + \int_a^b \psi_n^2(x)\,dx,$$

so that

$$\int_a^b \psi_n^2(x)\,dx = -2\int_a^b \varphi_n(x)\,\psi_n(x)\,dx.$$

Similarly

$$\int_a^b \psi_m^2(x)\,dx = -2\int_a^b \varphi_m(x)\,\psi_m(x)\,dx.$$

Next, since

$$\int_a^b f_m(x) f_n(x)\,dx = \int_a^b \varphi_m(x)\,\varphi_n(x)\,dx + \int_a^b \varphi_m(x)\,\psi_n(x)\,dx$$
$$+\int_a^b \varphi_n(x)\psi_m(x)\,dx + \int_a^b \psi_m(x)\psi_n(x)\,dx,$$

we have

$$\int_a^b \psi_m(x)\,\psi_n(x)\,dx = -\left[\int_a^b \varphi_m(x)\,\psi_n(x)\,dx + \int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right].$$

Hence the Schwarz inequality

$$\left[\int_a^b \psi_m(x)\,\psi_n(x)\,dx\right]^2 \leqq \int_a^b \psi_m^2(x)\,dx \cdot \int_a^b \psi_n^2(x)\,dx$$

becomes

$$\left[\int_a^b \varphi_m(x)\,\psi_n(x)\,dx + \int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right]^2$$
$$\leqq 4\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \cdot \int_a^b \varphi_n(x)\,\psi_n(x)\,dx,$$

which may be written

$$\left[\int_a^b \varphi_m(x)\,\psi_n(x)\,dx - \int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right]^2$$
$$\leqq 4\left[\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \,.\int_a^b \varphi_n(x)\,\psi_n(x)\,dx\right.$$
$$\left.-\int_a^b \varphi_m(x)\,\psi_n(x)\,dx \,.\int_a^b \varphi_n(x)\,\psi_m(x)\,dx\right].$$

Therefore we have the identities:

$$\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \cdot \int_a^b \varphi_n(x)\,\psi_n(x)\,dx \geqq 0,$$

$$\int_a^b \varphi_m(x)\,\psi_m(x)\,dx \cdot \int_a^b \varphi_n(x)\,\psi_n(x)\,dx$$
$$-\int_a^b \varphi_m(x)\,\psi_n(x)\,dx \,.\int_a^b \varphi_n(x)\,\psi_m(x)\,dx \geqq 0,$$

the latter belonging to the type treated by Prof. M. Fujiwara (1).

It follows from these inequalities, by a short calculation, that *there exist the following integral inequalities among* $f_m(x), f_n(x)$, $\varphi_m(x)$, $\varphi_n(x)$:

$$1 - \int_a^b f_m(x)\,\varphi_m(x)\,dx - \int_a^b f_n(x)\,\varphi_n(x)\,dx$$

(1) Fujiwara, Ein von Brunn vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der Schwarzschen und der Tchebycheffschen Ungleichungen für bestimmte Integrale, Tôhoku Math. Journal, 13 (1918), p. 228.

$$+\int_a^b f_m(x)\varphi_m(x)\,dx\cdot\int_a^b f_n(x)\varphi_n(x)\,dx \geqq 0,$$

$$1-\int_a^b f_m(x)\,\varphi_m(x)\,dx-\int_a^b f_n(x)\,\varphi_n(x)\,dx$$

$$+\int_a^b f_m(x)\,\varphi_m(x)\,dx\cdot\int_a^b f_n(x)\,\varphi_n(x)\,dx$$

$$-\int_a^b f_m(x)\,\varphi_n(x)\,dx\cdot\int_a^b f_n(x)\,\varphi_m(x)\,dx \geqq 0;$$

$$(m \neq n;\ m, n=1, 2, \ldots.)$$

which may be written respectively

$$\left[1-\int_a^b f_m(x)\,\varphi_m(x)\,dx\right]\left[1-\int_a^b f_n(x)\,\varphi_n(x)\,dx\right] \geqq 0\ (^1);$$

$$1-\int_a^b [f_m(x)\,\varphi_m(x)+f_n(x)\,\varphi_n(x)]\,dx$$

$$+\frac{1}{2}\int_a^b\int_a^b \begin{vmatrix} f_m(x), & f_m(y) \\ f_n(x), & f_n(y) \end{vmatrix} \cdot \begin{vmatrix} \varphi_m(x), & \varphi_m(y) \\ \varphi_n(x), & \varphi_n(y) \end{vmatrix}\,dx\,dy \geqq 0,$$

$$(m \neq n;\ m, n=1, 2, \cdots\cdots).$$

Ikeda near Ôsaka, May 1918.

---

(1) This is an immediate consequence of the Schwarz inequality: for from

$$\left[\int_a^b f_n(x)\,\varphi_n(x)\,dx\right]^2 \leqq \int_a^b f_n^2(x)\,dx\cdot\int_a^b \varphi_n^2(x)\,dx=1$$

we have

$$-1 \leqq \int_a^b f_n(x)\,\varphi_n(x)\,dx \leqq 1.$$

Determination of the Central Forces acting on a Particle whose Equations of Motion possess an Integral Quadratic in the Velocities,

by

Kinnosuke Ogura, Ôsaka.

Darboux determined the *conservative* forces acting on a particle whose equations of motion possess an integral (other than the integral of energy) of the form

$$(1) \qquad P\dot{x}^2+Q\,\dot{x}\,\dot{y}+R\dot{y}^2+S\dot{x}+T\dot{y}+K=\text{const.},$$

where P, Q, R, S, T, K are functions of the position of the particle (x, y) [1]. In this note I will treat the problem of similar nature for the *central* forces.

Let the equations of motion of a particle which is free to move under the central force F be

$$(2) \qquad \ddot{x}=-\frac{x}{\sqrt{x^2+y^2}}F(x,y), \quad \ddot{y}=-\frac{y}{\sqrt{x^2+y^2}}F(x,y).$$

It is required to find the function $F(x, y)$ in order that the differential equations (2) may possess an integral of the form (1), other than the integral of angular momentum.

Differentiating equation (1), and substituting for $\ddot{x}$ and $\ddot{y}$ from (2), we have

$$(3) \qquad \frac{\partial P}{\partial x}\dot{x}^3+\left(\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right)\dot{x}^2\dot{y}+\left(\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial x}\right)\dot{x}\,\dot{y}^2+\frac{\partial R}{\partial y}\dot{y}^3$$
$$+\frac{\partial S}{\partial x}\dot{x}^2+\left(\frac{\partial S}{\partial y}+\frac{\partial T}{\partial x}\right)\dot{x}\,\dot{y}+\frac{\partial T}{\partial y}\dot{y}^2$$
$$+\frac{\partial K}{\partial x}\dot{x}+\frac{\partial K}{\partial y}\dot{y}$$

(1) Darboux, Archieves Néerlandaises, (2) 6 (1901), p. 371; Whittaker, Treatise on the analytical dynamics (2. ed., 1917), p. 332.

$$-\frac{F}{\sqrt{x^2+y^2}}[2Px\dot{x}+Q(x\dot{y}+y\dot{x})+2Ry\dot{y}+Sx+Ty]=0.$$

Equating to zero the term of the third degree in $\dot{x}$ and $\dot{y}$, we have

$$\frac{\partial P}{\partial x}=0,\quad \frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}=0,\quad \frac{\partial Q}{\partial y}+\frac{\partial R}{\partial x}=0,\quad \frac{\partial R}{\partial y}=0,$$

from which it is easily seen that the terms of the second degree in the integral (1) must have the form

$$(ay^2+by+c)\dot{x}^2+(-2axy-bx-b'y+c_1)\dot{x}\dot{y}+(ax^2+b'x+c')\dot{y}^2,$$

where a, b, c, b', c', c_1 are constants.

Again equating to zero the terms of the second degree in $\dot{x}$ and $\dot{y}$ in (3),

$$\frac{\partial S}{\partial x}=0,\quad \frac{\partial S}{\partial y}+\frac{\partial T}{\partial x}=0,\quad \frac{\partial T}{\partial y}=0,$$

from which we have

$$S=my+p,\quad T=-mx+q,$$

where m, p, q are constants.

Further equating to zero the terms independent of $\dot{x}$ and $\dot{y}$ in (3), we have

$$Sx+Ty=0,$$

i. e.

$$px+qy=0.$$

This equation shows us that if p and q be different from zero, the force has the constant direction; for

$$\frac{\ddot{y}}{\ddot{x}}=\frac{y}{x}=-\frac{p}{q}.$$

Hence the constants p and q must each be zero.

Lastly equating to zero the terms of the first degree in $\dot{x}$ and $\dot{y}$ in (3),

$$\frac{\partial K}{\partial x}=\frac{F}{\sqrt{x^2+y^2}}(2Px+Qy),$$

$$\frac{\partial K}{\partial y}=\frac{F}{\sqrt{x^2+y^2}}(Qx+2Ry).$$

Differentiating the former with respect to y, and the latter with respect to x, and equating the two values of $\frac{\partial^2 K}{\partial x \partial y}$ thus obtained, we have

$$(2Px+Qy)\frac{\partial}{\partial y}\frac{F}{\sqrt{x^2+y^2}}+\left(2x\frac{\partial P}{\partial y}+y\frac{\partial Q}{\partial y}\right)\frac{F}{\sqrt{x^2+y^2}}$$

$$=(Qx+2Ry)\frac{\partial}{\partial x}\frac{F}{\sqrt{x^2+y^2}}+\left(x\frac{\partial Q}{\partial x}+2y\frac{\partial R}{\partial x}\right)\frac{F}{\sqrt{x^2+y^2}},$$

and replacing P, Q, R by their values as found above, we obtain

$$(4) \qquad (c_1 x+2c'y-bx^2+b'xy)\frac{\partial}{\partial x}\log\frac{F}{\sqrt{x^2+y^2}}$$
$$+(-2cx-c_1 y-bxy-b'y^2)\frac{\partial}{\partial y}\log\frac{F}{\sqrt{x^2+y^2}}$$
$$=3(bx-b'y).$$

This partial differential equation can be integrated in the following way. The characteristics are determined by

$$\frac{dx}{c_1 x+2c'y-bx^2+b'xy}=\frac{dy}{-2cx-c_1 y-bxy+b'y^2}=\frac{d\log F/\sqrt{x^2+y^2}}{3(bx-b'y)}$$

so that putting

$$\frac{F}{\sqrt{x^2+y^2}}=\frac{1}{z^3}$$

and

$$L=c_1 x+2c'y, \quad M=-2cx-c_1 y, \quad N=bx-b'y,$$

we get

$$(5) \qquad \frac{dx}{Nx-L}=\frac{dy}{Ny-M}=\frac{dz}{Nz},$$

which belongs to the type treated by Fouret (1). Consequently it follows that if we regard x, y, z as the rectangular point coordinates in space, *this system represents (special) W-curves in space.*

Now since

---

(1) Fouret, Comptes Rendus, Paris (1876); Wilczynski, Projective differential geometry of curves and ruled surfaces (1906), p. 282.

$$\frac{dx}{Nx-L}=\frac{dy}{Ny-M}$$

is a Jacobi equation, it can be integrated by the ordinary method, and the integral is

$$u^{2\lambda}\,v^{-\lambda}\,w^{-\lambda}=\text{const.},$$

i. e.

$$\frac{u^2}{v\,w}=\text{const.},$$

where

$$u\equiv\alpha\,x+\beta\,y-1,\ \ v\equiv 2\,c\,x+(c_1-\lambda)\,y,\ \ w\equiv 2\,c\,x+(c_1+\lambda)\,y\,;$$
$$\alpha\equiv(b\,c_1+2\,b'\,c)\,\lambda^{-2},\ \ \beta\equiv(b'\,c_1+2\,b\,c')\,\lambda^{-2},\ \ \lambda\equiv\sqrt{c_1^{\,2}-4\,c\,c'}\,.$$

Next in virtue of the identity

$$\alpha\,L+\beta\,M=N,$$

we obtain from (5)

$$\frac{\alpha\,dx+\beta\,dy}{\alpha\,x+\beta\,y-1}=\frac{d\,z}{z}\,,$$

whose integral is

$$\frac{z}{u}=\text{const.}\,.$$

Therefore the general integral of (4) is

$$F=\frac{\sqrt{x^2+y^2}}{u^3}\,\Psi\left(\frac{u^2}{v\,w}\right), \tag{6}$$

where Ψ is an arbitrary function. Thus we have arrived at the theorem:

The only cases of the motion of a particle, under the action of the central forces, which possess an integral quadratic in the velocities other than the integral of angular momentum, are those for which the force has the form

$$F=\frac{\sqrt{x^2+y^2}}{u^3}\,\Psi\left(\frac{u^2}{v\,w}\right),$$

where Ψ is an arbitrary function, and then the integral has the form

$$(a y^2+b y+c)\,\dot{x}^2+(-2\,a\,x\,y-b\,x-b'\,y+c_1)\,\dot{x}\,\dot{y}+(a\,x^2+b'\,x+c')\,\dot{y}^2$$
$$+m\,(y\,\dot{x}-x\,\dot{y})+K\,(x,y)=\text{const. } (^1),$$

$K\,(x, y)$ *standing for the integral*

$$\int \frac{\Psi}{u^3}\,[(N\,y-M)\,dx-(N\,x-L)\,dy]\,.$$

Here we add some particular cases:

I. If we put

$$b=b'=0,\quad c=c'=\frac{1}{2},\quad c_1=0,$$

then

$$F=-r\,\Psi\left(\frac{1}{r^2}\right),\quad (r=\sqrt{x^2+y^2})\,;$$

and the first integral becomes

$$\frac{1}{2}\,(\dot{x}^2+\dot{y}^2)-\int \Psi\left(\frac{1}{r^2}\right) r\,dr=\text{const.},$$

which is nothing but the equation of energy.

II. If we put

$$\Psi\,(\xi)=\text{const.},$$

then

(7) $$F=\frac{r}{(A\,x+B\,y+C)^3}\,,$$

A, B, C being arbitrary constants.

Again, if we put

$$\Psi\,(\xi)=\xi^{\frac{3}{2}}\,,$$

we have

(8) $$F=\frac{r}{(A_1 x^2+B_1 xy+C_1 y^2)^{\frac{3}{2}}}$$

A_1, B_1, C_1 being arbitrary constants.

---

(1) If we use the integral of angular momentum

$$y\,\dot{x}-x\,\dot{y}=k,\quad (k,\text{ any constant}),$$

the above equation may be written

$$c\,\dot{x}^2+c_1\,\dot{x}\,\dot{y}+c'\,\dot{y}^2+k\,(b\,\dot{x}-b'\,\dot{y})+K\,(x,y)=\text{const.}\,.$$

Now remembering that (7) and (8) give the laws of force discovered by Darboux and Halphen in the Bertrand problem [1], we infer the theorem:

When a particle describes a conic for any initial condition under a central force, the equations of motion have an integral quadratic in the velocities.

Ikeda near Osaka, May 1918.

---

(1) Darboux, Comptes Rendus, Paris, 84 (1877), p. 936; Halphen, ibid., p. 939; Appell, Traité de mécanique rationnelle, t. 1 (3. éd., 1909), p. 400.

抄　錄　短　評

I. 新　刊　書　目

Algrin. Eléments de mécanique générale et appliquée. Paris, Librairie de l'École spéciale des Travaux publics, 1917. 533 p.

É. Borel. Leçons sur les fonctions monogènes uniformes d'une variable complexe. Rédigées par G. Julia. (Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Émile Borel.) Paris, Gauthier-Villars, 1917. 12+156 p. Fr. 7.50.

E. A. Bawser. College algebra for the use of academies, colleges and scientific schools. 12th edition. Boston, Heath, 1917. 14+540 p.

R. Descartes. Œuvres de Descartes publiés par C. Adam et P. Tannery sous les auspices du Ministère du l'instruction publique. Paris, Cerf. Tomes 1–11: Œuvres, 1893–1909. Tome 12: Vie et œuvres de Descartes, étude historique par C. Adam, 1910. 20+646 p. [Tome 13]: Supplément, index générale, 1913. 108 p. Fr. 310.00.

Léon Lecornu. Cours de Mécanique, professé à l'École polytechnique t. III, Paris, 1918 Gauthier-Villars. Fr. 25.

近藤楠一著，新案加減法と餘對數の癈止，附算術減法の仕方を論ず．東京，大川屋書店，大正六年 1917, 61 頁．40 錢．

著者ハ栃木縣師範學校敎諭ニシテ頗ル計算法ノ改良ニ熱心ナリ．本書ノ發刊ニ續キテ「學生用乘除表」ヲモ出版セラレントス．計算界ノ感謝スベキトコロニシテ，又一顧チ此新案ニ向ケ改良ヲ講ズルノ楽トナスベシ．

佐野榮治著，理論應用最新實用數學．東京，丸善株式會社，大正七年 1918. 19+650+7 頁．4 圓．

本書分チテ三十六章トナス．乘法及除法ノ省略法，對數ノ理論，計算尺，三角法，三角形ノ解法，和角ノ三角函數，公式ノ用法，雜方程式恒算式及二項定理，級數，函數ノ圖示，實驗ノ結果ニ從ヒ法則ヲ定ムルコト，平均値及面積體積ノ測定，座標幾何學ノ大意，增加率，微分法，小ナル補正ノ計算法，累次微分，函數ノ展開，不定形，一ツノ變數ヲ有スル函數ノ極大及極小，二ツノ變數ヨリナレル函數，平面曲線ノ切線及法線，平面曲線ノ曲率半徑，擺線外擺線及內擺線，積分，定積分，平面曲線ノ面積，曲線ノ長サ，廻轉體ノ曲面積，立體ノ體積，廻轉體ノ體積，積分ノ平均値，平面形ノ中心及重心，慣性能率或第二能率，數學ノ力學的應用，應用物理學ニ關スル微分方程式，是レナリ．又特ニ加フルニ有用ナル定數表，三角函數ノ眞數表，數ノ對數表數ノ逆對數表，自然對數表，數ノ逆數，平方根，立方根，平方及立方ノ表，種々ノ材料ノ重量表，數學的物理的定數ノ表(種々ノ平面斷面ノ性質，種々ノ立體ノ性質)ガ加ヘラレ，又每章許多ノ練習問題ヲ以テ終ル．余ガ本誌前卷第 331 頁ニ於テ Rose 氏ノ著ヲ紹介スル際ニ述ベタルガ如ク，本書ノ出版ノ如キハ實ニ應急ノ良案タリ．此ノ時局ニ當リテ數學智識ノ頒布ハ眞ニ要用ナリト云ハザルベカラズ．高等數學ノ應用範圍ハ頗ル廣汎ナルモノニシテ，工學，物理學等ハ更ナリ，經濟學，統計學，保險學等ニ至ルマデ夫々數學ノ理論ヲ活用スルコソ其ノ道ノ發

達トナルモノナリ．精密ナル諸般ノ研究ノ基礎ハ數學ニ於テ見出サルベク，數學者ノ探索シ得タルトコロノモノハ理論ガ理論タルニ止マラズ其ノ應用ヲ見出シ得テ所謂有用ノ用タルナリ．余ハ理論ト應用トノ接近ヲ希ヒテ止マズ，本書ハ其慾ヲ充タスノ一端タランカ．(T. H.)

II. 雜誌內容

下記ノ雜誌ニ揭載セラレタル論文中，數學マタハ數理物理學ニ關係ナキモノハ省略ス．

The science reports of the Tôhoku Imperial University, (1) Vol, 6, No. 5, April, 1918.

M. Fujiwara, Zur Theorie der Differentialungleichungen, II. S. Kakeya, Theory of analytic functions of lines. T. Kubota, Über die (2, 2) deutigen quadratischen Verwandtschaften, I.

Tôkyô Sûgaku-Butsurigakkwai Kizi (Proceedings of the Tôkyô Math.-Physical Society, (2) Vol. 4, No. 12–15. Jan.—April, 1918.

S. Kakeya, On differential inequalities. K. Ogura, On the striped net of curves without ambages in dynamics. K. Ogura, On a generalization of the Bonnet-Darboux theorem concerning the line of striction.

Proceedings of the Edinburgh Mathematical Society, Vol. 35, Part 2, Nov., 1917.

H. Datta, On the failure of Heilermann's theorem. L. R. Ford, (1) A geometrical proof of a theorem of Hurwitz. (2) The effect of a rise in prices upon the amount of small money used. G. D. Taylor, Geometrical illustrations of cyclant substitutions. A. W. Young, On the computation of a Lagrangian interpolation.

The Messenger of Mathematics. Vol. 47, No. 5–7, 1917.

L. J. Mordell, On the class number for definite ternary quadratics. W. Burnside, On rational approximations to $\log x$. G. H. Hardy, Notes on some points in the integral calculus (47). H. Hilton and D. S. Tuck, On plane quartic curves with a tacnode. W. L. Glaisher, Table of binomial-theorem coefficients. G. N. Watson, A theorem of Liouville's type.

Proceedings of the Royal Academy of Amsterdam, Vol. 20, No. 3–5, 1918.

J. C. Kluyver, On hyperelliptic integrals of deficiency $p=2$, reducible by a transformation of order $r=4$. H. B. A. Bockwinkel, Some considerations on complete transformation. Jan. de Vries, Surfaces that may be represented in a plane by a linear congruence of rays. B. P. Haalmeyer, On elementary surfaces of the third order.

American Journal of Mathematics, Vol. 40, No. 2, April, 1918.

R. D. Carmichael, On the representation of functions in series of the form $\Sigma C_n g(x+n)$. L. P. Eisenhart, Transformations of planar nets. O. D. Kellogg, Orthogonal function sets arising from integral equations. C. H. Rawlins, Complete systems of concomitants of the three-point and the four-point in elementary geometry. A. L. Miller, Systems of pencils of lines in ordinary space. T. Dantzig, Some con-

tributions to the geometry of plane transformations. P. Sperry, Properties of a certain projectively defined two-parameter family of curves on a general surface.

Transactions of the American Mathematical Society, Vol. 19, No. 1, Jan., 1918.

L. R. Ford, Rational approximations to irrational complex numbers. H. B. Mitchell, On the imaginary roots of a polynomial and the real roots of its derivative. M. Fréchet, Relations entre les notions de limite et de distance. A, D. Pitcher and E. W. Chittenden, On the foundations of the calcul fonctionnel of Fréchet. H. J. Ettlinger, Existence theorems for the general real self-ajoint linear system of the second order. T. H. Hildebrandt, On the boundary value problems in linear differential equations in general analysis. O. E. Glenn, A fundamental system of formal covariants modulo 2 of the binary cubic.

Bulletin of the American Mathematical Society, Vol. 24, No. 7, April, 1918.

C. J. Keyser, The rôle of the concept of infinity in the work of Lucretius. A Emch, On the invariant net of cubics in the Steinerian transformation. T. Fort, Some theorems of comparison and oscillation. W. L. Hart, Note on infinite systems of linear equations. L. D. Cummings, An undervalued Kirkman paper. F. Cajori, Pierre Laurent Wantzel. R. D. Carmichael, Additive functions of a point set.

The American Mathematical Monthly, Vol. 25, No. 1–3, 1918.

E. V. Huntington, Bibliographical notes on the use of the word "mass" in current text books. W. D. Cairns, Third annual meeting of the Mathematical Association of America. D. E. Smith, and J. Ginsburg, Rabbi Ben Ezra and the Hindu-Arabic problem. R. A. Johnson, The theory of similar figures. W. H. Metzler, Note on a certain class of determinants.

Annales scientifiques de l'École Normale Supérieure, (3) T. 34, No. 8–12, 1917.

A. Denjoy, Mémoire sur la totalisation des nombres dérivés non sommables. Garnier, Étude de l'intégrale générale de l'équation VI de M. Painlevé dans le voisinage de ses singularités transcendantes. É. Picard, Sur une équation fonctionelle se présentant dans la théorie de la distribution de l'électricité avec la loi de Neumann.

Bulletin des Sciences Mathématiques, (2) T. 42, Fév., 1918

G. D. Birkhoff, Sur la démonstration directe de dernier théorème de Henri Poincaré par M. Dantzig. H. Villat, Quelques r cents progrès des théories hydrodynamiques.

Nouvelles Annales de Mathématiques, (4) T. 18, Fév., 1918.

A. P. Appell, Groupes de points sur l'hyperbole équilatère: exercice proposé. É Turrière, Au sujet d'un article de M. A. Gérardin.

Annali di Matematica pura ed applicata, (3) T. 27, Fasc. 1–2, 1918.

Bordiga, Sul modello minimo della varietà delle n-ple non ordinate dei punti di un piano. A. Palatini, Sulla meccanica delle verghe. C. Segre, Sui complessi lineari di piani nello spazio a cinque dimensioni. P. Tortorici, Nuovi studi sulle superficie rigate

Atti della Reale Accademia dei Lincei, (5) Vol. 27, Fasc. 1–3, (1 Semestre), 1918.

T. Levi-Civita, ds^2 einsteiniani in campi newtoniani. II: Condizioni di integrabilità e comportamento geometrico spaziale. G. Ricci, Sulle varietà a tre dimension dotate di terne principali di congruenze geodetiche. O. Tedone, Sulle ovali di Cartesio

come curve aplanetiche di rifrazione. A. Del Re, Hamiltoniani e gradienti di hamiltoniani e di gradienti laplassiani parametri differenziali. V. Vesin, Proprietà del prodotto graduale. G. Armellini, Ricerche sopra la previsione dell'urto nel problema dei tre corpi. C. Burali-Forti, Differenziali esatti. U. Cisotti, Una formola per la determinazione di dislivelli dei dorsi d'acqua mediante misure di velocità. G. Sannia, Sulle serie di potenze di una variabile sommate col metodo di Borel generalizzato. C. Burali-Forti, Alcune linee e superficie collegate con una linea gobba. A. Pensa, Una espressione differenziale vettoriale alternata.

Atti della Reale Accademia delle Scienze di Torino, Vol. 53, Disp. 3–4, 1917–1918.

C. Burali-Forti, Alcuni sistemi di linee su di una superficie. E. Boverio, Sopra la derivazione dei canali. G. Sannia, Le serie di potenze di una variabile sommate col metodo di Borel generalizzato (Nota I, II). C. Guidi, Sul calcolo dell'arco elastico senza cerniere. A. G. Rossi, Un transformatore dinamico per correnti alternate (Nota IV).

Bulletin de l'Académie des Sciences de Russie, (6) No. 5–6, 1918.

A. A. Markov, Généralisation du problème de l'échange successif des boules.

雜 錄 彙 報

歐 米 諸 大 學 課 程

瑞 西

ば ー ぜ る 大 學 (1917–18 夏 學 期)

へっけ E. Hecke, 微分積分學 (4), 演習 (1), 數論 (1), すぴーす Spiess 教授ト共同ノせみなりー (1). すぴーす, 解析幾何學 (3), 數學ノ基本觀念 (4), 數學史 (1), へっけ教授ト共同ノせみなりー (1). ふらっと R. Flatt, せみなりー (3), 射影幾何學 (2), くなっぷ M. Knapp, 天體物理學 (2), 一般測時論 (1), 通俗天文學, 月 (1), 天文學演習 (3), 同ジク初學者ニ對スルモノ (2). まちいす W. Matthies, 數理物理學演習 (2).

べ る ん 大 學 (1917–18 夏 學 期)

ぐらーふ Graf, 球函數 (4), べっせる函數 (5), 微分學 (3), 函數論 (2), 微分方程式 (2), 利息算及び保險數學 (2), ふーばー教授ト共同ノ數學演習 ($1\frac{1}{2}$). ふーばー Huber, 天體力學 (2), 代數表面 (3), ふーりえ級數ト其應用 (2), ぐらーふ教授ト共同ノせみなりー (1). おっとー Otto, 代數解析 (2), 球面三角法及び其應用 (2), 積分學 (1), 解析幾何學 (2). べんてり Benteli, 畫法幾何學, 曲線, 線織面, 正多面體 (2), 畫法幾何學演習 (2), 實用幾何 (1). まうでるり Mauderli, 天文學數理地理學ニ於ケル教育上ノ諸問題 ($1\frac{1}{2}$), 地質學及ビ研究旅行者ニ對スル天文學的併ビニ風土學的位置決定法 (1), 最近ノ發達ヨリ見タル宇宙ノ構造 ($1\frac{1}{2}$). ふーばー, 自性(ナチュラル)幾何學 (1), 平面三次曲線 (1). くれりえ Crelier, 綜合幾何學 (1), n 次空間ノ幾何學, べるりーなー Berliner, 高等代數學 ($1\frac{1}{2}$). もーざー Moser, 寡婦孤兒扶助金ノ數學的研究, 保險數學ニ關スル特選題目 ($1\frac{1}{2}$), 保險數學演習 ($1\frac{1}{2}$). ぼーれん Bohren, 統計學 (2), 最小自乘法 (2).

ふらいぶるぐ大學 (1917–18, 夏學期)

ぷらんしゅれる Plancherel 微積分學 (4) 演習 (1), 高等代數學 (3). だにえるす Daniëls 解析幾何學 (4), 演習 (2), 解析重學 (4), 函數論 (3).

ぜねゔぁ大學 (1917–18, 夏學期)

けーらー Cailler, 微分積分學 (3), 演習 (2), 理論重學 (3), 演習 (2), 解析學ニ關スル談話, 解析函數論 (2). ふぇーる Fehr, 高等數學初步 (3), 代數學幾何學ノ補充 (1), 演習 (1), 射影幾何學 (1), 高等幾何學ニ關スル談話 (2), 初等數學演習, 方法論及ビ教授法ニ關スル特選題目 (1). ごーちぇ R. Gautier, 數理天文學汎論 (2), 氣候學 (2). ちーあしー G. Tiercy, 殞外彈道學. 其他私講師ノ講義トシテハ次ノモノアリ. べるぬー Bernoud, 圖解法 (曲線及ビ計算機) (1). ちーあしー, 直交網線ノ理論 (1).

ろーざんぬ大學 (1917–18 夏學期)

あむすたいん Amstein, 函數論 (3), 積分學補充 (2). ぢゅーま G. Dumas, 微積分學 (6), 演習 (2), 數學演習 (1). らこむべ Lacombe, 畫法幾何學 (4), 模型 (4), 解析幾何學 (2), 位置幾何學及ビ演習 (3). めーやー Mayor, 理論重學 (4), 演習 (1), 數理物理學 (2). めーらーる Maillard, 極微解析及ビ其科學上ノ應用 (4), 球面天文學 (3), 理論重學 (2). ぢゅーま S. Dumas, 確ラシサノ理論 (3), 其私講師ノ講義トシテぢあこてっと Ch. Jacottet, 函數論ノ特選題目 (2), ぱしょど Paschoud, 數理物理學諸論 (2).

にうしやてる大學 (1917-18, 夏學期)

ぱすきえ Pasquier, 微積分學 (3), 微分方程式, 變換群 (2), 數學演習 (2), 保險數學緒論 (1). かべれる L. Gaberel, 解析幾何學 (2). 解析函數論 (2). る ぐらん ろわ Le Grand Roy, 球面天文學 (2). 演習 (1). 氣象學 (1). 天文學 (特選題目) (1). ぢゃけろー A. Jaquerod, 理論重學 (2). 私講師ノ講義トシテハすとれーる H. Stroele, 最小自乘法及ビ誤差論 (1). あるんと L. Arndt, 天體物理學諸論 (1).

ちゅーりひ大學 (1917–18, 夏學期)

ふゅーたー Fueter, 自然科學ノ數學的取扱ヒ (3), 演習 (1), 函數論 (3). すぱいざー教授ト共同ノせみなりー (1). すぱいざー Speiser, 微積分學 (4), 演習 (1). 射影幾何學 (3), 積分ノ觀念 (1). べるないす Bernays, 三角級數論 (3). うぉるふぁー Wolfer, 天文學諸論 (2), 演習 (2), 蝕ノ理論 (2)

ちゅーりひ市立工業學校 (1917–18, 夏學期)

ひるしゅ Hirsch, 高等數學 (6), 演習 (2). ふらねる Franel, 高等數學 (6), 演習 (2). ぐろすまん Grossmann, 畫法幾何學 (4), 演習 (2), 射影幾何學 (4). わいる Weyl, 解析幾何學 (2), 演習 (2). こるろす Kollros, 畫法幾何學 (4), 演習 (1), 位置幾何學 (3), 演習 (1). まいすなー Meissner, 重學 (4), 演習 (2). ふーるゐっつ Hurwitz, 代數方程式 (4). ふーるゐっつ, わいる, せみなりー (2). わいる, 幾何學特選題目 (4), 數學ノ論理的基礎 (1). まいすなー, 重學特選題目 (2). べしゅりん Baeschlin, 測量學, 高等測地學 (3). うぉるふぁー, 天文學諸論 (3). 演習 (2), 蝕ノ理論 (2). あむべるぐ Amberg, 恩給保險數學 (2). ぶらんでんべるがー Brandenberger, 數學自然科學教育緒論 (2). ぽるや Pólya, 確ラシサ及ビ統計論 (2). 隨意科トシテあ

むべるぐ, 保險ノ問題 (1). ばいえる Beyel, 計算尺, 演習 (1). 畫法幾何學 (2). 射影幾何學 (1). ごんせーと Gonseth, ばいらしょなる變換 (2). 圖式計算 (2). けらー Keller, 平面及ビ空間相稱系及ビ其二次曲線二次曲面ニ對スル應用 (2). きーなすと Kienast, ぺっせる函數 (2). くらふと Kraft, 世界ノ源力 (1). 幾何學的解析 (3). 幾何的解析ニヨル可變形物體ノ重學 (3). ぽるや Pólya, 數學遊戯 (1).

正三角形ノ一極小性質

定長 h ノ線分ガ内部デ一廻轉シ得ル凸閉曲線ノ最小面積ヲ有スルモノハ, 恐クハ高サ h ノ正三角形ナラントハ吾等ノ想像シ得ル所デアルガ, 此ノ想像ガ事實ナリトノ證明ハ未ダニ得ラレナイ. 玆ニハ唯カヽル性質ヲ有スル三角形並ニ凸四邊形ノ内丈ケデ高サ h ノ正三角形ガ最小面積ヲ有スルコトノ證明ヲ摘記スル.

1. 任意ノ凸四邊形 $ABCD$ ヲ取ル. 定長 h ノ線分ガソノ内デ一廻轉シ得ル爲メニハ, 何レノ邊ヲ取リテモソノ邊上ニナキ二頂點カラノ距離ノ大ナル方ガ $\geqq h$ トナラナケレバナラナイ.

2. 頂點 C ヲ適當ニ選ブト C ヨリ AB, AD ヘ下シタ垂線ガ, ソレゾレ D, B ヨリ AB, AD ヘ下シタ垂線ヨリ大ナラヌ樣ニスルコトガ出來ル. 此ノ場合 A カラ BC, CD ヘ下シタ垂線ハ必ラズ $\geqq h$ デナケレバナラヌ.

3. A ヲ中心トシ半徑 h ノ圓 (C) ヲ作ル. AB, AD ヲ二邊トシ他ノ二邊ハ圓 (C) ニ切スル凸四邊形 $AB'C'D'$ ヲ作リ角 $AB'C'$, $AD'C'$ ヲシテ角 A ト等シカラシムレバ(但シ $\angle A<\frac{\pi}{2}$ トス) B', D' ハソレゾレ BA, DA 上ニ落チル. 故ニ $B'C'$ ガ圓 (C) ニ切スル點ヲ M. N トスレバ線分 AB', $B'M$, ND', $D'A$ 及ビ (C) 圓ノ弧 MN ヨリナル凸形 (K) ノ面積ハモトノ四邊形 $ABCD$ ノソレヨリ小デアル.

4. 此ノ (K) ノ面積ヲ角 A ノ函數ト考ヘテ之ヲ極小ナラシムルト, ソハ A ガ 60 度トナリ M, N ガ一致シテ (K) ガ高サ h ノ正三角形トナルコトヲ解析的ニ證明シ得ラレル, 但シ凸形ナルコトヲ始終念頭ニオイテ論ズルヲ要スル.

5. $\angle A\geqq\frac{\pi}{2}$ ノ場合ハ $B'C'$, $D'C'$ ヲシテソレゾレ AD, AB ニ平行ナラシメテ (K) ヲ作レバ (K) ノ面積ハ $\angle A$ ト共ニ小トナルガ故ニ $\angle A<\frac{\pi}{2}$ ノ場合ニ歸着セシメラル.

6. カクシテ高サ h ノ正三角形ハ $ABCD$ ヨリ小ナルコトガ證セラレタ. 三角形同志ノ内デハ勿論正三角形ガ所設ノ條件ノ下デハ最小デアル, 故ニ上記ノ定理ガ成立スル. (M.F.)

森吉太郎氏ノ根函數

慶應義塾大學講師森吉太郎氏ハ方程式ノ解法ノ研究ニ熱心ナリ. 同氏ハ其ノ所謂根函數ナルモノヲ用ヒテ其ノ目的ヲ遂ゲントスルモノナリ. 之ニ關スル同氏ノ論述ハ東京物理學校雜誌第 11 卷明治三十五年 1902, 第 15 卷明治三十九年 1906, 第 19 卷明治四十三年 1910 等ニアリ. 根函數トハ函數 $f(\alpha, x)$ ヲ x ノ冪ニ展開シテ

$$f_0(\alpha)x^{m_0}+f_1(\alpha)x^{m_1}+f_2(\alpha)x^{m_2}+\cdots f_{n-1}(x)x^{m_{n-1}}$$

トシ各係數中ノ α ヲシテ其項ノ x ノ指數ニ等シカラシムルモノ卽チ

$$f_0(m_0)x^{m_0}+f_1(m_1)x^{m_1}+f_2(m_2)x^{m_2}+\cdots f_{n-1}(m_{n-1})x^{m_{n-1}}$$

ナリ. 此結果ヲ又 $f(\alpha\ x)$ ノ値ト云ヒ $S[f(\alpha, x)]$ ニテ表ハス. (委細前記雜誌ヲ見ルベシ). 最近ノ通知ニ依ルニ同氏ハ次ノ定理ヲ得タリト.

$$F(D)y=S\left[\frac{1}{\alpha!}f(x)\right] \quad \text{ナラバ} \quad y=S\left[\frac{1}{\alpha!}\cdot\frac{f(x)}{F(x^{-1})}\right] \quad \text{ナリ.}$$

證明.

今 $$y=S\left[\frac{1}{\alpha!}f(x)\right]$$ トオクトキハ

$$\frac{dy}{dx}=\frac{d}{dx}S\left[\frac{1}{\alpha!}f(x)\right].$$

然ルニ微係數ヲ取ルハ α ヲカケテ x ニテ割ルニ等シ, 故ニ

$$\frac{dy}{dx}=x^{-1}S\left[\frac{1}{(\alpha-1)!}f(x)\right]=S\left[\frac{1}{\alpha!}f(x)x^{-1}\right].$$

(x^{-1} ヲ [ ] ノ內ニ入ルルトキハ α ハ 1 ヲ增スベキニヨル)

同樣ニ

$$\frac{d^2y}{dx^2}=S\left[\frac{1}{\alpha!}f(x)\,x^{-2}\right],$$

$$\frac{d^3y}{dx^3}=S\left[\frac{1}{\alpha!}f(x)\,x^{-3}\right],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

此ノ各ニ $A_0, A_1, A_2, \cdots$ ヲ掛ケテ加フルトキハ

$$A_0y+A_1\frac{dy}{dx}+A_2\frac{d^2y}{dx^2}\cdots\cdots\cdots\cdots\cdots\cdots$$

$$=S\left[\frac{1}{\alpha!}f(x)\,(A_0+A_1x^{-1}+A_2x^{-2}+\cdots\cdots\cdots\cdots)\right],$$

但シ　$A_0, A_1, A_2, \cdots$　ハ x, y ヲ含マズ

今 $f(x)$ ニ代フルニ　$f(x)/(A_0+A_1x^{-1}+A_2x^{-2}+\cdots\cdots\cdots)$ ヲ以テスルトキハ次ノ如ク述ブルコトヲ得

$$A_0y+A_1\frac{dy}{dx}+A_2\frac{d^2y}{dx^2}+\cdots\cdots\cdots \quad =S\left[\frac{1}{\alpha!}f(x)\right] \quad \text{ナルトキハ}$$

$$y=S\left[\frac{1}{\alpha}\cdot f(x)/(A_0+A_1\,x^{-1}+A_2\,x^{-2}+\cdots\cdots\cdots)\right]$$

ナリ. 之レヲ書キ直シテ

$$F(D)y=S\left[\frac{1}{\alpha!}f(x)\right]$$

ナルトキハ

$$y=S\left[\frac{1}{\alpha!}\cdot f(x)/F(x^{-1})\right]$$

ナリトスルコトヲ得.

例.　$y+\dfrac{dy}{dx}=x^n/n!$　ヲ解ケ (n ヲ完全數トス, 但シ一般ニハ n ニ制限ナシ).

$$x^n/n!=S\left[\frac{1}{\alpha!}x^n\right]$$

又

$$y+\frac{dy}{dx}=\left(1+\frac{d}{dx}\right)y,$$

故ニ　$f(x)=x^n$,　$F(D)=1+D$,

故ニ

$$y=S\left[\frac{1}{\alpha!}\cdot x^n/(1+x^{-1})\right]=S\left[\frac{1}{\alpha!}\cdot x^{n+1}(1+x)\right]$$

$$=S\left[\frac{1}{\alpha!}(x^{n+1}-x^{n+2}+x^{n+3}-\cdots\cdots\cdots)\right].$$

$$=\frac{x^{n+1}}{n+1\,!}-\frac{x^{n+2}}{+n2\,!}+\frac{x^{n+3}}{n+3\,!}-\cdots\cdots\cdots\cdots$$

$$=e^{-x}-\left(1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\cdots\cdots-\frac{x^n}{n!}\right),\qquad (n\ \text{奇數})$$

又ハ

$$=1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\cdots\cdots+\frac{x^n}{n!}-e^{-x},\qquad (n\ \text{偶數}).$$

然ルニ茲ニ $\pm e^{-x}$ ニ對シテハ

$$\pm e^{-x}+\frac{d}{dx}(\pm e^{-x})=0,$$

故ニ

$$y=-\left(1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+\cdots\cdots-\frac{x^n}{n\,!}\right),\qquad (n\ \text{奇數})$$

又ハ

$$y=1-x+\frac{x^2}{2\,!}-\frac{x^3}{3\,!}+-\cdots\cdots+\frac{x^n}{n\,!}.\qquad (n\ \text{偶數})$$

擬心曲線ニ就テ

一ツノ平面的閉曲線ノ內部ニ一點アリテ之ヲ通過シ任意ノ方向ニ引キタル此曲線ノ弦ノ二ツノ部分ノ和ガ一定ナルトキハ，此點ヲ此曲線ノ擬心 Pseudo-centre ト名ヅケ其曲線ヲ擬心曲線 Pseudo-central curve ト名ヅクルコトハ余ガ先キニ東北帝國大學理科報告第五卷，1916, 第 303 頁ニ於テ提議セシコトナリ．斯ノ如キ曲線ハ其擬心ヲ極點トセル極式坐標 (r,θ) ニヨリテ表ハサレタルトキ

$$r(\theta)+r(\pi+\theta)=2a$$

ナル性質ヲ有ス． コハ藤原教授ニ原ヅク思想ナルガ，之ヲ更ニ擴張シテ

$$r(\theta)+r(\theta+\alpha)+r(\theta+2\alpha)+\cdots\cdots+r(\theta+\overline{m-1}\alpha)=ma,$$

$$\alpha=\frac{2\pi}{m}$$

ナル性質ヲ有スル曲線ヲ考ヘ得テ其極式方程式ヲ決定シ得ルコトヲ同報告同卷第312頁ニ揭ゲタリ． カヽル理想的ニ案出シタル曲線モ亦實用ヲ見ルコトアルモノト見エ The American Mathematical Monthly, 第 24 卷 1917, 第 290 頁ニハ R. P. Baker 氏ノ提出セル次ノ問題アリ．

A designer of machinery requires a curve having the following properties:

(1) A closed curve touching a given circle at two diametral points and enclosing it.

(2) The sum of the three radii from the center of this circle to the curve which make with each other angles of 120° is constant.

(3) The locus of a point which lies at some constant distance from the curve on its inner normal must be such that it is also the locus of a point fixed on a bar of some simple linkage. In estimating the value of the word "simple" pivoted bars are preferred to slides and the total number should be as small as possible.

Condition (3) is needed to enable a cylinder to be ground accurately to the curve.

Tobias Dantzig 氏之ニ答ヘタルガ其第二條件ハ卽チ前記ノ方程式ニ於テ $m=3$ トセル場合ナリ． 數學ノ硏究ニ從事スル者ノ理想ガ期セズシテ應用ヲ得ルノ一例ナルベシ (T. H.)

An Example in Differential Geometry.

The total curvature of the surface, whose linear element is

$$ds^2 = v^m du^2 + u^m dv^2,$$

is equal to

$$\frac{2-m}{m}\left(\frac{1}{\rho_{gu}^2}+\frac{1}{\rho_{gv}^2}\right),$$

where ρ_{gu} and ρ_{gv} are the radii of geodesic curvature of the parametric curves v=const. and u=const. respectively.

Hence the surface is

of negative curvature when $m<0$,
of positive curvature when $0<m<2$,
and of negative curvature when $m>2$.

It is to be noticed that when $m=1$, the total curvature is just equal to the sum of the squares of the geodesic curvatures of the parametric curves (T. H.)

A Characteristic Property of the Pseudosphere.

The surface of revolution whose geodesic curvatures along its parallels have constant value is the pseudosphere; and conversely. The radius of geodesic curvature is equal to the radius of total curvature (T. H.)

二三雜誌中ノ注目スベキ論說記事

保險醫學雜誌第八十號
邦人ノ命數ニ就イテ　理學士 鈴木敏一氏
保險雜誌, 大正7年3月號及5月號
年金ノ利率ヲ求ムル公式(承前)　門脇政治氏
年金ノ利率ヲ求ムル方法ニ就テ　理學士 竹下清松氏
東京物理學校雜誌, 大正7年5月號及6月號
ぱすかるノ定理及其擴張ニ就テ　理學博士 林鶴一氏
非ゆーくりっど幾何學ノぽあんかれー, くらいんノ表示
理學博士 窪田忠彥氏

夏季講習會

京都帝國大學ニテハ第九回講演會ヲ本年1918八月一日ヨリ開ク. 數學ニ關スルモノハ次ノ如シ

微積分學概念, (自八月一日至同十日自午前七時至同九時)
理科大學講師 松本敏三氏

又文部省ノ依囑ヲ受ケテ仙臺理科大學ニ開カルルモノハ七月二十五日ニ始リ十日間續キ次ノ如シ

微積分及ビ微分方程式ノ概要 (每日一時間半ヅヽ)
理科大學助教授理學博士 掛谷宗一氏

數學ノ應用方面概觀 (每日一時間半ヅヽ)
初等數學 (幾何學ヲ除キタル) ノ歷史(每日一時間ヅヽ)
同教授 理學博士 藤原松三郎氏

廣島高等師範學校及東京女子高等師範學校ニ開カルル分ハ次ノ如シ

(廣島) 自七月二十五日至八月七日
超限數 (二十四時)　同校教授 菅禮太郎氏

中等學校ニ於ケル數學敎科及其ノ敎授法ニ就テ(十六時)

同校講師 波木井九十郎氏

幾何學敎授上ノ諸問題 同校敎授 角 達助氏

中等敎育數學敎授ニ關スル講習員ノ研究發表及意見交換(二時間ヅヽ凡五回)

同校講師 波木井九十郎氏
同校敎授 角 達助氏

(東京) 自七月二十五日至八月七日(兩科併セテ四十八時間)

日常諸算ニ就テ 同校敎授 森 岩太郎氏

初等平面幾何學作圖法 同校講師 牧田らく氏

諸學者ノ消息

佛蘭西巴里學士院ニテハぱんるゔゑー敎授 Paul Painlevé ヲ président ニ選擧シけーにぐ氏 Gabriel Kœnigs ヲろーてー氏 H. Leauté ノ代リニ力學科委員ニ選擧セリ

瑞西べるん大學くるりえー氏 L. Crelier ハ同大學ノ敎授トナレリ.

獨逸國ぼん大學ノはーん氏 H. Hahn ハ同大學ノ敎授トナリべっく氏 A. Beck ハ助敎授トナレリ.

獨逸國ぶれすらう大學ノ敎授しゅみっと氏 E. Schmidt ハべるりん大學ノ敎授トナレリ. 獨逸國かーるするーえ高等工業學校ノぢすてり敎授 Disteli ハ病氣ノ爲隱退シくらうすたーる鑛山學校敎授タリシもーあまん氏 H. Mohrmann 之レニ代レリ.

墺太利ゔゐーん高等工業學校ニテハすつっとがると高等工業學校敎授めーむけ R. Mehmke みゅんへん高等工業學校敎授ふゐんすてるわるだー Finsterwalder ノ兩氏ニ honorare Doktoren ノ學位ヲ授ゲタリ.

佛蘭西りおん大學敎授おゥとんぬ氏 Léon Autonne ハ 1916 年 1 月 12 日 57 歲ニテ逝去セリ.

那威國はんねゔゐぐ氏 C. Hannevig ハあーべるノ紀念資金トシテ 150,000 Crowns ヲ提出セリ之レソノ利子ヲ以テ那威國ニ於ケル特種ノ數學研究者ニ授與セシガ爲メナリト.

獨逸國ぼん大學ノ敎授ろんどん氏 Franz London ハ 1917 年 2 月 17 日五十四歲ニテ逝去セリ. はむぶるがー氏 Alfred Hamburger ハろんどん敎授ノ記念資金トシテ 30,000 Mark ヲ大學ニ寄附シタリ. 北米合衆國おはよーはいでるばーぐ大學ノほーにんぐ氏 Christian Horning ハ 1918 年 1 月 31 日 73 歲ニテ逝去シわしんとん大學數學及畫法幾何學敎授タリシえんぐらー氏 Engler ハ 1918 年 1 月 16 日 61 歲ニテ逝去セリ.

北米合衆國じおるだん及すとぅふぁー M. F. Jordan, Q. Stauffer ノ兩氏ハめーいん大學ノ數學敎師トナレリ.

墺太利ゔゐーんノ學士院ハあいんすたいん敎授ニばうむがるてん賞牌ヲ授與セリ.

寺尾壽, 千本福隆, 坂井英太郎, 高木貞治, 國枝元治, 吉江琢兒, 中川銓吉, 森岩太郎, 黑田稔, 阿部八代太郎, 渡邊孫一郎ノ諸氏ハ大正 7 年 6 月 13 日附ヲ以テ敎員檢定委員會臨時委員被仰付.

Sets of Independent Postulates concerning Equality and Inequality (Theory of Three Undefined Relations),

by

KUNIZÔ YONEYAMA, Kumamoto.

CONTENTS.

PART I.

FUNDAMENTAL SETS OF INDEPENDENT POSTULATES.

PART II.

SINGULAR SETS OF INDEPENDENT POSTULATES.

PART III.

CATEGORICAL SETS OF INDEPENDENT POSTULATES.

Introduction.

The notion of equality and inequality being one of the fundamental conceptions of mathematics, we find several sets of axioms concerning them in mathematical works. Dr. Otto Stolz, in his "Theoretische Arithmetik," states that the following conditions must always be satisfied whenever equality and inequality are defined.

1. Es muss jedes Ding sich selbst gleich sein.
2. Wenn $a=b$ ist, so muss eben deswegen $b=a$ sein. Desgleichen wenn a und b ungleich sind, so auch b und a.
3. Je zwei Dinge des Systems müssen entweder gleich oder ungleich sein.
4. Wenn $a=b$ und $b=c$ ist, so muss $a=c$ sein.

And he also states the following as the necessary conditions to be satisfied by the definitions of "die grössere und die kleinere Grösse".

5. Wenn $a>b$ ist, so muss $b<a$ sein und umgekehrt.
6. Von je zwei ungleichen Grössen muss die eine als die grössere, die andere als die kleinere erklärt sein.
7. Wenn $a=b$, $b>c$, so muss $a>c$ sein.
8. Wenn $a>b$, $b>c$, so muss $a>c$ sein.

From these propositions he deduces other propositions concerning equality and inequality.

Bertrand Russell in his "Principles of Mathematics" states the following as the indemonstrable axioms which the relational theory of magnitude obliges us to assume.

a. $A=B$, or A is greater than B, or A is less than B.
b. A being given, there is always a B, which may be identical with A, such that $A=B$.
c. If $A=B$, then $B=A$.
d. If $A=B$ and $B=C$, then $A=C$.

e. If A is greater than B, then B is less than A.

f. If A is greater than B, and B is greater than C, then A is greater than C.

g. If A is greater than B, and $B=C$, then A is greater than C.

h. If $A=B$ and B is greater than C, then A is greater than C.

From these propositions he also deduces all the other propositions. But neither Stolz nor Russell gives any complete proof of their mutual independence([1]).

E. V. Huntington, in his "The Continuum" and in "Annals of Mathematics", series 2, vols. 6 and 7, gives a set of assumptions concerning the relation $<$ as follows.

1. If a and b are distinct elements of K, then either $a<b$ or $b<a$.
2. If $a<b$, then a and b are distinct.
3. If $a<b$ and $b<c$, then $a<c$.

Here he gives the proofs of their mutual independence. But this set of assumptions concerns only one relation $<$.

In the following paper, I have tried to construct sets of axioms concerning equality and inequality, which have the three fundamental properties required in a set of axioms, namely independence, consistency and sufficiency of axioms.

In Part I, I have constructed fundamental sets of independent axioms which are sufficient to deduce all propositions concerning equality and inequality, and further to construct as many of them as possible and to compare their characteristic features. Also when investigating their properties, I have found that a very slight modification of one of the axioms sometimes led to a set of axioms presenting very remarkable properties quite different from what we shall expect. Thus I have been led to construct a more general set of axioms, containing the two main branches, one of which is the usual, while the other is quite strange. These are discussed in Part II.

Moreover it is true that the classes of things, in which all the propositions concerning equality and inequality hold, cannot necessarily be brought into a one-to-one correspondence, retaining their mutual relations of equality and inequality. In other words, the sets of Part I are not categorical. Therefore, in Part III, I have constructed the categorical sets of independent postulates concerning equality and inequality.

([1]) In his book above cited, p. 100, Stolz also mentions 6 (but 7 in substance) axioms concerning equality and inequality, and only adds that every axiom is independent of the preceding ones, but the proofs of their mutual independence are not given.

PART I.

Fundamental Sets of Independent Postulates.

Of several sets of independent axioms obtained by the suitable combination of propositions concerning equality and inequality, we select two of them as representative, each of them having certain characteristics. After having proved that each of these sets has the three fundamental properties, we shall proceed to discuss generally all possible sets and their mutual relations.

SECTION 1. SETS OF INDEPENDENT POSTULATES.

THE FIRST SET OF POSTULATES.

We will take a class of things, the number of whose elements may be finite or infinite. The equality or inequality of these elements to each other are the relations between them, defined by a set of the following five propositions.

I. *Any two elements of the class satisfy at least one of the three relations "equal to", "greater than" and "less than".*

II. *Any element of the class has at least one element which is equal to it.*

III. *If A is less than B, then B is greater than A.*

IV. *If A is equal to B and B is greater than C, then A is greater than C.*

V. *If A is greater than B and B is greater than C, then A is greater than C.*

This set of axioms is characterised by the fact that the mutual relation of axioms is very delicate and even the slightest alteration of an axiom of the set, which may seem to be almost negligible causes an essential change in the properties of the set. For example, if we replace Axiom II by an analogous one II′:

"*Any element has at least one element, to which it is equal,*"

then the new set of axioms is quite different from the original, for in a class of things satisfying the new set of axioms, there may be two elements A, B, such that they satisfy the following propositions, which are very different from the usual.

1. If A is equal to B, then B is greater than A.
2. A is greater than itself.

Similarly when we replace Axiom III by an analogous one:

"*If A is greater than B, then B is less than A*",

a similar change in the properties of the set occurs. So this set of axioms leads us to inquire into some singular sets of axioms. In Part II, we shall discuss these subjects in more detail.

This set of axioms is also characterised by the variety both in the form and in the nature of every axiom in the set, compared with the second set of axioms. Namely, Axiom I asserts that any two elements ought to be in a certain relation to each other, and Axiom II asserts that for any element of the class there must exist an element having a definite relation to it. Axioms III, IV, V, are propositions in usual form, and the hypothesis of Axiom III implies only one relation while that of Axiom IV implies two relations of different kinds and that of Axiom V two relations of the same kind. But it is to be remarked that, in contrast with the variety of their hypothesis, their conclusions are all the same and are concerned with only one relation, viz. "greater than".

THE SECOND SET OF POSTULATES

I. Any two elements of the class satisfy at least one of the three relations "equal to", "greater than", and "less than".

II. If A is greater than B, then B is less than A.

III. If A is less than B, then B is greater than A.

IV. If A is equal to B and B is equal to C, then A is equal to C.

V. If A is greater than B and B is greater than C, then A is greater than C.

This set of axioms is more symmetrical and elegant in its form than the first, though they are identical in substance; and the nature of the mutual relations of the axioms is very different from that of the first; namely if we replace Axiom V by the analogous one V′:

"*If A is less than B and B is less than C, then A is less than C*",

we get a set of axioms having the same property as the original, and if either Axiom II or III is replaced by the other (analogous in nature and form) the new set is again identical with the old (there are no Axioms analogous to I and IV).

From these, we may affirm that one of the characteristic properties of the second set of axioms is the stableness of the mutual relations of its axioms while that of the first set of axioms is their unstableness. In the last part of this section, we shall have occasion to add another remarkable fact explaining these characteristics of both sets.

Moreover, in this set, the form and nature of the axioms are not so different as those of the first set, for Axioms II, III are very similar in form and nature; and Axioms IV, V have also very regular forms, the three relations contained in their hypothesis and conclusions being all the same. But, on the other hand, it is to be noted that the conclusions of the axioms of the second set include all the cases of the three relations, while the conclusions of the axioms of the first set include only one relation, viz. "greater than."

In the following, we shall show that these two sets of axioms have the three fundamental properties required in a set of axioms. But, in doing so, when we have to see what propositions will follow from a set of axioms by logical deduction, we must be careful to insure that our deduction is accurate. The only way to insure this is to think of our fundamental propositions, not as axiomatic propositions about things, but as blank forms in which $A, B, C \cdots\cdots$ may donote any objects we please, and the relations "equal to", "greater than", "less than" any relations we please. The deductions made from such blank forms must necessarily be purely formal, and so will not be affected by the troublesome connotations which often attach themselves to any concrete interpretations of the symbols. Therefore we shall treat the above set of axioms as a blank form, namely as a set of postulates rather than as axioms. For that reason, it will be better to write the above postulates in symbols as follows.

The first set of postulates.

Postulate I. *Any two elements A, B of the class satisfy at least one of the three relations $A \textcircled{$=$} B$, $A \textcircled{$>$} B$, $A \textcircled{$<$} B$.*

Postulate II. *Any element A has at least one element B, such that $B \textcircled{$=$} A$.*

Postulate III. *If $A \textcircled{$<$} B$, then $B \textcircled{$>$} A$.*

Postulate IV. *If $A \textcircled{$=$} B$ and $B \textcircled{$>$} C$, then $A \textcircled{$>$} C$.*

Postulate V. *If $A \textcircled{$>$} B$ and $B \textcircled{$>$} C$, then $A \textcircled{$>$} C$.*

The second set of postulates.

Postulate I. *Any two elements A, B of the class satisfy at least one of the three relations $A \textcircled{$=$} B$, $A \textcircled{$>$} B$, $A \textcircled{$<$} B$.*

Postulate II. *If $A \textcircled{$>$} B$, then $B \textcircled{$<$} A$.*

Postulate III. *If $A \textcircled{$<$} B$, then $B \textcircled{$>$} A$.*

Postulate IV. *If* $A \circleddash B$ *and* $B \circleddash C$, *then* $A \circleddash C$.

Postulate V. *If* $A \circledgtr B$ *and* $B \circledgtr C$, *then* $A \circledgtr C$.

From this point of view, our work becomes much more general than a study of the ordinary relations of equality and inequality; it is a study of any system which satisfies the conditions laid down in our postulates, and so it constitutes an abstract theory of the three undefined relations $\circleddash$, $\circledgtr$, $\circledless$.

In the study of independent postulates, it is very important to pay attention to the meaning of the words in which the postulates are expressed. Misunderstanding often occurs from a vague notion of the postulates used. It will not therefore be out of place to explain clearly the meaning of our set of postulates.

I. By our Postulate III "if $A \circledless B$, then $B \circledgtr A$", we mean that from the hypothesis $A \circledless B$, only one relation $B \circledgtr A$ follows, and the other relations $B \circleddash A$ and $B \circledless A$ do not occur with $B \circledgtr A$ at the same time. This being the natural meaning our postulate is to be understood accordingly throughout this paper unless specially stated otherwise. That is to say, the postulate "if $A \circledless B$, then $B \circledgtr A$" is to be understood as the abbreviation of the proposition "if two elements A, B of the class satisfy the relation $A \circledless B$, then B, A always satisfy the relation $B \circledgtr A$, but never the relations $B \circleddash A$ and $B \circledless A$".

But, on the other hand, as the postulate only asserts that, from the hypothesis $A \circledless B$, the relation $B \circledgtr A$ always follows; and as, besides this assertion, it says nothing as to whether the other relations concur or not with the relation $B \circledgtr A$; and moreover as nothing hinders the concurrence of these relations since our relations $\circleddash$, $\circledgtr$, $\circledless$ are pure symbols and have no meaning in themselves, one may suppose that the conclusion of the postulate may sometimes accompany the other relations $B \circleddash A$ or $B \circledless A$ with the relation $B \circledgtr A$ (1).

The same may be said of Postulates IV and V. The difference of the above meanings, though it seems very trifling, has a considerable effect in the establishment of a set of independent postulates. If the meaning of our postulates be taken as the former, then from each of the first and second sets of postulates we can prove that any two elements A, B of the class satisfy only one of the three relations $A \circleddash B$, $A \circledgtr B$, $A \circledless B$, (the proof of this will be given in section 4). But if the meaning of our postulates be taken as the latter, then the above proposition cannot

(1) Of course we may construct a class of things having such a property.

be deduced from our set of postulates, so in this case Postulate I must be replaced by Postulate I′ :—

Postulate I′. Any two elements A, B of the class satisfy one, and only one, of the three relations $A \ominus B$, $A \gtrdot B$, $A \lessdot B$.

The proof of the above will be given in section 2. But, if, in the first set of postulates, the conclusion of only one Postulate IV is admitted to be unique, then the proposition in question may be deduced from the first set of postulates. And also in the second set the same may be said when the conclusions of Postulates II and III are admitted to be unique. In the latter case, the uniqueness of the conclusion of only one of Postulates II and III is insufficient for the purpose. Its proof will be given in section 2; and as to the deduction of the proposition in question in each case, it will be given in section 4.

II. In the propositions concerning equality and inequality, one often uses the words "it is equal to itself" or "A is equal to A". In this paper also such words are often used. Now, as equality and inequality are the relations between two elements of a class of things considered, if, in that class, a certain element is repeated, or, in other words, if the class contains two or more elements A's, which are distinct, but are interchangeable with one another throughout a given discussion then the proposition "A is equal to A" may have the ordinary meaning the former A and the latter A being considered as distinct. But if the class of things contains only one element A, how shall we interpret the above proposition? Here we may choose between the following two.

Firstly, we may reject the above proposition as absolutely meaningless in such a class of things; and secondly, considering the image of the element A in our mind, or considering the element A as consisting of two coincident elements A and A, we may understand the above proposition as the relation between these two elements.

If we take the first interpretation, the proposition "if $A \ominus B$ and $B \ominus C$, then $A \ominus C$" has a meaning when, and only when, A and C are distinct, while, according to the second interpretation, it has a meaning even when A and C are one and the same element. Throughout this paper, we shall take the second interpretation unless specially stated otherwise.

The interpretations of the above proposition also give a considerable effect to the set of independent postulates, for, if we take the first interpretation, then the proposition:

"any two elements A, B of the class satisfy only one of the three

relations $A ⊜ B$, $A ⧁ B$, $A ⧀ B$,"

cannot be deduced from our set of postulates, even when their meaning is taken as the former one stated in I (the proof of this will be given in section 2). Moreover when we take the first interpretation, in order that our set of postulates may be sufficient to deduce all the propositions concerning equality and inequality from it, one more postulate

"Postulate VI. Every element of the class is repeated at least once" must be added. For, without this postulate we cannot prove the proposition "if $A ⊜ B$, then $B ⊜ A$" and many others from our set of postulates (the proof of this will be given in section 2). Therefore the complete set of postulates in this case is as follows.

The set (A).

I′. Any two elements A, B of the class satisfy one and only one of the three relations $A ⊜ B$, $A ⧁ B$, $A ⧀ B$.

II. Any element A of the class has at least one element B, such that they satisfy the relation $B ⊜ A$.

III. If $A ⧀ B$, then $B ⧁ A$.

IV. If $A ⊜ B$ and $B ⧁ C$, then $A ⧁ C$.

V. If $A ⧁ B$ and $B ⧁ C$, then $A ⧁ C$.

VI. Any element A of the class is repeated at least once.

But, here by the addition of Postulate VI, if the uniquenss of the conclusion of the postulate be admitted, then the latter part of Postulate I′ becomes superfluous and we may therefore omit it. Nevertheless if it be not admitted, then the whole of Postulate I′ must be preserved and each postulate is independent of the other.

Further we may replace Postulates II and VI by Postulate II′:

"Any element A of the class has at least one repeated element A satisfying the relation $A ⊜ A$."

Thus in this case we have the following set of postulates.

The set (B).

I. Any two elements A, B of the class satisfy at least one of the three relations $A ⊜ B$, $A ⧁ B$, $A ⧀ B$.

II′. Any element A has at least one repeated element A, such that they satisfy the relation $A ⊜ A$.

III. If $A ⧀ B$, then $B ⧁ A$.

IV. If $A ⊜ B$, and $B ⧁ C$, then $A ⧁ C$.

V. If $A ⧁ B$ and $B ⧁ C$, then $A ⧁ C$.

That the postulates of this set are independent and sufficient will be proved in sections 2 and 4. It is to be noted that Postulate II′ is not

equivalent to Postulates II and VI since Postulate II′ cannot be deduced from Postulates II and VI (the proof of this will be given in section 2). But, strangely enough when each of them is combined with the same Postulates I, III, IV, V, they form equivalent sets of postulates, for, the set (A) is equivalent to the set (B), each of them being deduced from the other.

Next compare the first set and the set (B), and notice that they are exactly the same except the difference of only one letter, namely that the letter B of Postulate II is replaced by the letter A of Postulate II′. When the relation of an element to itself is admitted, from both of these sets all the propositions concerning equality and inequality can be deduced. But when it is not admitted, a striking difference arises between the two. For, any class of things satisfying the set (B) always satisfies all the postulates of the first set, while the converse of it is not necessarily true. In other words, if we consider the class of all classes of things satisfying the postulates of the first set, then it contains not only all classes satisfying the postulates of the set (B), but also other classes which do not satisfy them. Among the latter classes, there are some in which one or more of the following five fundamental propositions concerning equality and inequality do not hold good, while all of them necessarily must do so in all classes satisfying the postulates of the set (B).

Proposition I. If $A \textcircled{$=$} B$, then $B \textcircled{$=$} A$.
Proposition II. If $A \textcircled{$>$} B$, then $B \textcircled{$<$} A$.
Proposition III. If $A \textcircled{$=$} B$ and $B \textcircled{$=$} C$, then $A \textcircled{$=$} C$.
Proposition IV. If $A \textcircled{$>$} B$ and $B \textcircled{$=$} C$, then $A \textcircled{$>$} C$.
Proposition V. If $A \textcircled{$<$} B$ and $B \textcircled{$=$} C$, then $A \textcircled{$<$} C$.

The above fact shows that these five propositions cannot be deduced from the first set when the relation of an element to itself is not admitted, while all of these and all others are always deduced from the set (B), (the proof of this will be given in section 9). *By this example we shall see how great an influence the difference of only one letter may have on a set of postulates.*

Thus when both the uniqueness of the conclusions of the postulates and the relation of an element to itself are admitted, the first set is a sufficient set of independent postulates concerning equality and inequality; and when the former is admitted while the latter is not, the set (B) is a sufficient set of independent postulates concerning it; and when the latter is admitted while the former is not, or when both of them are not admitted, in order to have sufficient sets of independent postulates, we

have only to add the condition that any two elements of the class must satisfy only one of the three relations to postulate I of the first set and the set (B) respectively. Thus, from this point of view, we have the four different sets of postulates, and, for convenience of reference, we shall call them the sets of the first type, the second type, the third type and the fourth type respectively, and recapitulate them here in full.

Set of the first type.

I. *Any two elements A, B of a class of things satisfy at least one of the three relations $A \textcircled{=} B$, $A \textcircled{>} B$, $A \textcircled{<} B$.*

II. *Any element A has at least one element B, satisfying the relation $B \textcircled{=} A$.*

III. *If $A \textcircled{<} B$, then $B \textcircled{>} A$.*

IV. *If $A \textcircled{=} B$ and $B \textcircled{>} C$, then $A \textcircled{>} C$.*

V. *If $A \textcircled{>} B$ and $B \textcircled{>} C$, then $A \textcircled{>} C$.*

Set of the second type.

I. *The same as that of the first type.*

II′. *Any element A has at least one repeated element A, satisfying the relation $A \textcircled{=} A$.*

III, IV, V. *The same as those of the first type.*

Set of the third type.

I′. *Any two elements A, B of a class of things satisfy one and only one of the three relations $A \textcircled{=} B$, $A \textcircled{>} B$, $A \textcircled{<} B$.*

II, III, IV, V. *The same as those of the first type.*

Set of the fourth type.

I′. *The same as that of the third type.*

II′. *The same as that of the second type.*

III, IV, V. *The same as those of the first type.*

Hitherto we have dealt with the first set of postulates only. But if the second set be taken and the above considerations be applied to it, we find here again a remarkable difference in the characters of the first and the second sets. Namely, when the relation of an element to itself is not admitted, the first set not only loses the proposition $A \textcircled{=} A$, but also the five fundamental propositions concerning equaltiy and inequality as we have already seen. On the other hand, the second set loses only one proposition $A \textcircled{=} A$ as its necessary consequence, and all others are left unchanged. In other words, *the first set is very delicate and unstable and changes its essential properties by a very slight modification of it while the second set is stable and remains almost unchanged by such modification.*

SECTION 2. INDEPENDENCE OF THE POSTULATES.

The method of proving the independence of a proposition A from other propositions $B, C, \cdots\cdots, M$, usually consists in finding a class of things which satisfies the propositions $B, C, \cdots\cdots, M$, but not A. The legitimacy of this method is obvious, since the existence of such a class of things shows that the proposition A cannot be logically deduced from the propositions $B, C, \cdots\cdots, M$. Hence, in the following, we shall use this method and give classes of things having such a property (firstly number systems, and then other concrete systems of objects).

To construct number systems having the required property, we assume the existence of real numbers, and make the following conventions according to the common usage. When a, b denote two real numbers, we use the symbol "$a>b$" instead of saying $a-b$ is positive, and the symbol "$a<b$" instead of saying $a-b$ is negative, and the symbol "$a=b$" instead of saying $a-b$ is zero. Moreover, henceforth the words "equal to", "greater than" and "less than" are used in a sense quite different from the common usage, they are used to denote any relations whatever provided that they satisfy a given set of postulates, and they are used instead of the symbols $\circledequal$, $\circledgtr$, $\circledless$. We begin with the proof of the independence for the first set of postulates.

Independence of the postulates of the first set.

(A). Postulate I is independent of Postulates II, III, IV, V.

Consider a class of numbers $\{A(a, b)\}$, where a, b denote any real numbers, and suppose that every element of the class is repeated any number of times. Now define the relations of its elements as follows:—
(i). If $A(a, b)$ and $B(a', b')$ denote the two elements of the class, then A is said to be equal to B (or, in symbol, $A \circledequal B$) when, and only when, $a=a'$ and $b=b'$; and (ii) A is said to be greater than B (or, in symbol, $A \circledgtr B$) when, and only when, $a-a'>b-b'$; and (iii) A is said to be less than B (or, in symbol, $A \circledless B$) when, and only when, $a-a'<b-b'$. Then this class of numbers satisfies Postulates II, III, IV, V, but not Postulate I. (For the convenience of reference we shall call this class of numbers the Class ($\mathfrak{A}$).)

Proof. (a). The class does not satisfy Postulate I. For, select four real numbers a, a', b, b', so that $a \neq a'$, $b \neq b'$, $a-a'=b-b'$ (it is clear that such numbers exist in our class) and with these numbers construct two numbers $A(a, b)$, $B(a', b')$ of the class, then by the above definitions

we cannot determine whether A is equal to, or greater than, or less than B. Thus the class of numbers does not satisfy Postulate I.

(b). The class satisfies Postulate II. For, every element of the class is repeated any number of times, so if we denote two of them by $A_1(a, b)$, $A_2(a, b)$, then, by definition, A_1 is equal to A_2.

(c). The class satisfies Postulate III. For, when $A(a, b)$ is less than $B(a', b')$, we have by definition

$$a-a'<b-b',$$

therefore

$$a'-a>b'-b,$$

which shows that B is greater than A.

(d). The class satisfies Postulate IV. For, when $A(a, b)$ is equal to $B(a', b')$ and B is greater than $C(a'', b'')$, we have

$$a=a', b=b'; \qquad a'-a''>b'-b'';$$

therefore

$$a-a''>b-b'',$$

which shows that A is greater than C.

(e). The class satisfies Postulate V. This may be proved in exactly the same manner as in (d).

(B). Postulate II is independent of Postulates I, III, IV, V.

Consider a class of numbers $\{A(a, b)\}$, where a denotes one of the integers less than 100, and b denotes zero; and further add to this class a number $N(a=100, b=1)$. In this aggregate $\{A(a, b), N\}$, define the relations of its elements as follows:— (i). If $A(a, b)$ and $B(a', b')$ denote two elements of the class, then A is said to be equal to B when, and only when, $a-a'=b+b'$; and (ii) A is said to be greater than B when, and only when, $a-a'<b+b'$; and (iii) A is said to be less than B when, and only when, $a-a'>b+b'$. Then this class of numbers satisfies all the postulates other than Postulate II. (When every element of this class is repeated any number of times, we shall call this class the Class ($\mathfrak{B}$).)

Proof. (a) The class satisfies Postulate I. This is evident from the above definitions.

(b). The class does not satisfy Postulate II. Take $N(100, 1)$ as a given element A, then there is no element which is equal to A. For, in order that $B(a', b')$ should be equal to A, the relation $a'-100=b'+1$ must hold good, but, since $a'\leqq 100$ and $b'=1$ or 0, it clearly does not. Thus A has only no other element equal to it, but also it is not even equal to itself.

Remark 1. It is to be noted that N *has another element* $M(99, 0)$, *to which* N *is equal, but none which is equal to* N. *Moreover in this class* $\acute{N}$ *is greater than itself.*

Remark 2. In this class of numbers, any element has always an element to which it is equal, though a certain element has none which is equal to it.

(c). The class satisfies Postulate III. For, when $A(a, b)$ is less than $B(a', b')$, we have

$$a-a'>b+b', \qquad (b+b'=0, \text{ or } 1, \text{ or } 2),$$

and accordingly

$$a-a'>0,$$

and so

$$a'-a<0,$$

from which follows

$$a'-a<b'+b.$$

Thus when A is less than B, B is greater than A.

(d). The class satisfies Postulate IV. For, when A is equal to B and B is greater than $C(a'', b'')$, we have

$$a-a'=b+b',$$
$$a'-a''<b'+b'',$$

and accordingly

$$a-a''<b+b''+2b'. \tag{1}$$

But by the definition of equality, if A denote $N(100, 1)$, then B must be the number $M(99, 0)$; and if A denotes any number other than N, then B must be A itself. Therefore in any case b' must be zero, hence from (1) we have $a-a''<b+b''$, which shows that A is greater than C.

(e). The class satisfies Postulate V. For, when A is greater than B and B is greater than C, we have

$$a-a'<b+b',$$
$$a'-a''<b'+b'',$$

and accordingly

$$a-a''<b+b''+2b'. \tag{1}$$

(i). Now if A denote $N(100, 1)$, then B must also be N, since, if not, we would have the relation $100-a'<1+0$ (where $a' \lesseqqgtr 99$), which is clearly impossible. When B denotes N, C must also be N by the same reason. Thus in this case we have

$$a-a''=100-100, \qquad b+b''=1+1,$$

and accordingly

$$a-a''<b+b'',$$

which shows that A is greater than C.

(ii). If A is any element other than N, then B may or may not be N. First suppose that B is N, then C must also be N. Thus in this case

$$a-a''=a-100, \qquad b+b''=0+1,$$

therefore

$$a-a''<b+b'',$$

which shows that A is greater than C. Secondly suppose that B is not N, then $b'=0$ and from (1) we have at once

$$a-a''<b+b''$$

showing that again A is greater than C.

Remark. If we consider a class of all real numbers, every element except 1 being repeated any number of times and the element 1 not being repeated, and if we define equality and inequality of its elements in the usual manner, then this class of numbers (which we shall call the Class ($\mathfrak{B}'$)) will be sufficient to prove the independence of Postulate II from the other postulates, when the relation of an element to itself is not admitted. But as the remarkable property of the element $N(100, 1)$ of the previous class ($\mathfrak{B}$) is often used afterwards, and morover as the class may be always used, no matter whether the relation of an element to itself is admitted or not, here we use this class as an example to prove the independence of Postulate II.

(C). Postulate III is independent of Postulates I, II, IV, V.

Consider a class of numbers $\{A(a, b)\}$, where a denotes any real number and b any positive real number; and define the relations of its elements as follows :— (i). If $A(a, b)$ and $B(a', b')$ denote two elements of the class, then A is said to be equal to B when, and only when $a-a'=b-b'$; and (ii) A is said to be less than B when, and only when, $a-a'<b+b'$ and $a-a'\neq b-b'$; and (iii) A is said to be greater than B when, and only when, $a-a'\gtreqless b+b'$. According to these definitions, this class of numbers satisfies all the postulates other than Postulate III. (When every element of this class is repeated any number of times, we shall call this class of numbers the Class ($\mathfrak{C}$).).

Proof. (a). That this class satisfies Postulates I and II is evident from the above definitions.

(b). The class does not satisfy Postulate III. For, when a, a' are given we may find b, b', so that

$$|a-a'|<b+b', \qquad a-a' \neq b-b',$$

by taking b, b' sufficiently great. With the four numbers thus determined, construct two elements $A(a, b)$, $B(a', b')$ of the class, then since

$$a-a'<b+b' \qquad a-a' \neq b-b',$$

A is less than B by definition. But, since

$$|a-a'|<b+b',$$

we have

$$a'-a<b'+b, \qquad a'-a \neq b'-b,$$

which shows that B is also less than A. Thus in this class there are some elements which do not satisfy Postulate III.

(c). The class satisfies Postulate IV. For, when A is equal to B and B is greater than $C(a'', b'')$, we have

(1) $$a-a' = b-b',$$

(2) $$a'-a'' \geqq b'+b''.$$

By adding (1) to (2) we get the relation

(3) $$a-a'' \geqq b+b'',$$

which shows that A is greater than C.

(d). The class satisfies Postulate V. For, when A is greater than B and B is greater than C, we have

$$a-a' \geqq b+b',$$
$$a'-a'' \geqq b'+b'',$$

and accordingly

$$a-a'' \geqq b+b''+2b';$$

but by the hypothesis b' is positive, therefore we have

$$a-a''>b+b''.$$

Thus in this case A is greater than C.

(D). Postulate IV is independent of Postulates I, II, III, V.

Consider a class of numbers $\{A(a, b, c)\}$, a denoting a positive integer of the form $1+nd\ (d>3)$, and b, c any positive proper fractions; and define the relations of its elements as follows.

If $A(a, b, c)$ and $B(a', b', c')$ denote two elements of the class, then

A is said to be equal to B when, and only when, one of the following relations holds;

$$\text{I.}\quad\begin{cases}1.\ \ a-a'+b-b'=c-c'.\\ 2.\ \ a=a',\ b=b',\ c\neq c'.\\ 3.\ \ a=a',\ b\neq b',\ b-b'>c-c',\ c-c'\not>0.\\ 4.\ \ a=a',\ b\neq b',\ b-b'<c-c',\ c-c'\not<0.\end{cases}$$

The remaining cases are divided into the following two:

$$\text{II.}\quad\begin{cases}5.\ \ a-a'+b-b'>c-c',\ a\neq a'.\\ 6.\ \ a=a',\ b\neq b',\ b-b'<c-c',\ c-c'<0.\end{cases}$$

$$\text{III.}\quad\begin{cases}7.\ \ a-a'+b-b'<c-c',\ a\neq a'.\\ 8.\ \ a=a',\ b\neq b',\ b-b'>c-c',\ c-c'>0.\end{cases}$$

In the case II, A is said to be greater than B, and in the case III, A is said to be less than B. Then this class of numbers satisfies all the postulates other than Postulate IV. (When every element of this class is repeated any number of times, we shall call this class the Class ($\mathfrak{D}$).)

Proof. (a). The class satisfies Postulate I. For, any two elements A, B satisfy one, and only one, of the three relations:

(a). $a-a'+b-b'=c-c'$.

(b). $a-a'+b-b'>c-c'$.

(c). $a-a'+b-b'<c-c'$.

Now each of the relations (b) and (c) may be divided into the following three cases:

$$\begin{cases}(\mathrm{b}_1)\ \ a\neq a',\ a-a'+b-b'>c-c'.\\ (\mathrm{b}_2)\ \ a=a',\ b\neq b',\ a-a'+b-b'>c-c'.\\ (\mathrm{b}_3)\ \ a=a',\ b=b',\ a-a'+b-b'>c-c'.\end{cases}$$

$$\begin{cases}(\mathrm{c}_1)\ \ a\neq a',\ a-a'+b-b'<c-c'.\\ (\mathrm{c}_2)\ \ a=a',\ b\neq b',\ a-a'+b-b'<c-c'.\\ (\mathrm{c}_3)\ \ a=a',\ b=b',\ a-a'+b-b'<c-c'.\end{cases}$$

and further the relations (b_2) and (c_2) may be subdivided into the two cases:

$$\begin{cases}(\mathrm{b}_2')\ \ a=a',\ b\neq b',\ c-c'>0,\ a-a'+b-b'>c-c'.\\ (\mathrm{b}_2'')\ \ a=a',\ b\neq b',\ c-c'\not>0,\ a-a'+b-b'>c-c'.\end{cases}$$

$$\begin{cases}(\mathrm{c}_2')\ \ a=a',\ b\neq b',\ c-c'<0,\ a-a'+b-b'<c-c'.\\ (\mathrm{c}_2'')\ \ a=a',\ b\neq b',\ c-c'\not<0,\ a-a'+b-b'<c-c'.\end{cases}$$

Thus any two elements of the class satisfy one, and only one, of the relations (a); (b_1), (b_2'), (b_2''), (b_3); (c_1), (c_2'), (c_2''), (c_3). Now (a), (b_2''), (b_3), (c_2''), (c_3) belong to case I, and (b_1), (c_2') to case II, and (b_2'), (c_1) to case III. Therefore any two elements of the class satisfy one, and only one, of the relations of cases I, II, III.

(b). The class satisfies Postulate II. For, given any element $A(a, b, c)$ of the class, we can always find a', b', c', such that

$$a'=a, \quad b'=b, \quad c'\neq c,$$

and if we construct $B(a', b', c')$ with these numbers a', b', c', then by the definition I, 2, B is equal to A.

(c). The class satisfies Postulate III. This is evident from the definitions of "greater than" and "less than".

(d). The class does not satisfy Postulate IV. In this class there are elements which do not satisfy Postulate IV. For example, taking two positive proper fractions $b<\frac{88}{100}$, $c<\frac{87}{100}$, and an integer $a=1+nd$, select a', b', c'; a'', b'', c'', so that

$$a'=a, \quad b'=b, \quad c'=c+\frac{2}{100},$$

$$a''=a, \quad b''=b+\frac{2}{100}, \quad c''=c+\frac{3}{100},$$

then $A(a, b, c)$, $B(a', b', c')$ and $C(a'', b'', c'')$ are the elements of our class, and yet they do not satisfy Postulate IV.

In the first place, since

$$a=a', \quad b=b', \quad c\neq c',$$

by definition I, 2, A is equal to B. Next, since

$$a'=a'', \quad b'\neq b'', \quad b'-b''<c'-c''<0,$$

by definition II, 6, B is greater than C. Lastly since

$$a=a'', \quad b\neq b'', \quad b-b''>c-c'', \quad c'-c''<0,$$

by definition I, 3, A is equal to C. Thus while A is equal to B and B is greater than C, A is equal to C, and so they do not satisfy Postulate IV.

(e). The class satisfies Postulate V. For, when A is greater than B, and B is greater than C, we have

(i) $$a-a'+b-b'>c-c', \quad a\neq a',$$

or

(ii) $\quad a=a', \quad b\neq b', \quad b-b'<c-c', \quad c-c'<0;$

and

(iii) $\quad a'-a''+b'-b''>c'-c'', \quad a'\neq a'',$

or

(iv) $\quad a'=a'', \quad b'\neq b'', \quad b'-b''<c'-c'', \quad c'-c''<0.$

Accordingly here the four cases are to be distinguished.

Case I. Combination of (i) and (iii).

In this case $a-a''$ is not zero. For, if it were, $(a-a')+(a'-a'')$ would be zero and therefore $(a-a')$ and $(a'-a'')$ would be of opposite signs. But since

$$|a-a'|>3, \quad |b-b'|<1 \quad \text{and} \quad |c-c'|<1,$$

we have $a-a'>0$ from (i), and similarly $a'-a''>0$ from (iii). Thus $(a-a')$ and $(a'-a'')$ cannot be of opposite signs and accordingly $a-a''$ is not zero. Further by addition we have

$$a-a''+b-b''>c-c''$$

from (i) and (iii). So by definition II, 5, A is greater than C.

Case II. Combination of (ii) and (iv).

In this case we have

(ii) $\quad a=a', \quad b\neq b', \quad b-b'<c-c', \quad c-c'<0,$

and

(iv) $\quad a'=a'', \quad b'\neq b'', \quad b'-b''<c'-c'', \quad c'-c''<0;$

and from these relations by addition we have at once

$$a=a'', \quad b-b''<c-c'', \quad c-c''<0.$$

Further, since $c-c'<0$ and $b-b'<c-c'$, $b-b'$ is negative, and similarly $b'-b''$ is also negative. Therefore the sum of them must also be negative. Thus in this case we have

$$a=a'', \quad b\neq b'', \quad b-b''<c-c'', \quad c-c''<0,$$

which shows that A is greater than C.

Case III. Combination of (i) and (iv).

In this case we have

(i) $a \neq a', \quad a-a'+b-b'>c-c',$

(iv) $a'=a'', \quad b' \neq b'', \quad b'-b''<c'-c'', \quad c'-c''<0.$

From (i) we know that $a-a'$ is positive and by the hypothesis it is greater than 3; and moreover since $a'-a''$ is zero, we have

$$a-a''>3.$$

Thus the relation

$$a-a''+b-b''>c-c''$$

clearly holds good. Further from the relations $a \neq a'$, $a'=a''$, it follows that $a \neq a''$. So by definition II, 5, A is greater than C.

Case IV. Combination of (ii) and (iii).

This case may be treated in exactly the same manner as in case III.

Thus, in all cases, under the hypothesis that A is greater than B, and B is greater than C, A is always greater than C.

Remark. This example is somewhat complicated, but it serves also to prove the independence of Postulate IV in the second set of postulates. Therefore this example is given here. If we wish to get a class of numbers serviceable for the proof of the independence of Postulate IV in the first set of postulates only, we may have the following simpler one.

Consider a class of all rational numbers and define the relations of its elements as follows:— (i). If $A(a)$ and $B(b)$ denote two elements of the class, then A is said to be equal to B when, and only when, $a+1=b$; and (ii) A is said to be less than B when, and only when, $a<b$ and $a+1 \neq b$; and (iii) A is said to be greater than B when, and only when, $a \geqq b$. In this class every element may be repeated any number of times. We shall call this class of numbers the Class $(\mathfrak{D}')$. Now this class $(\mathfrak{D}')$ satisfies all the postulate other than Postulate IV. For, from the definitions of equality and inequality, it is clear that this class satisfies Postulates I, II, III, V. And when we take three rational numbers $A(a)$, $B(a+1)$, $C\left(a+\frac{1}{2}\right)$, it is at once seen that A is equal to B and B is greater than C, while A is less than C. Therefore these three numbers do not satisfy Postulate IV.

(E). Postulate V is independent of Postulates I, II, III, IV.

Consider a class of numbers $\{A(a, b, c)\}$, where a, c denote any real numbers and b any positive real number, and suppose that every element is repeated any number of times. Now, in this class, define the relations

(i). If $A(a, b, c)$ and $B(a', b', c')$ denote two elements of the class, then A is said to be equal to B when, and only when,

$$a=a', \quad b=b', \quad c=c';$$

and (ii) A is said to be greater than B when, and only when,

$$a-a'+b+b' \gtreqless c-c' \quad \text{(excluding the case } a=a',\ b=b',\ c=c');$$

and (iii) A is said to be less than B when, and only when,

$$a-a'+b+b'<c-c'.$$

Then this class of numbers satisfies all the postulates other than Postulate V. (We shall call this class of numbers the Class ($\mathfrak{C}$).)

Proof. (a). That this class satisfies Postulates I, II, IV is evident from the above definitions of equality and inequality.

(b). The class satisfies Postulate III. For, when A is less than B, we have

$$(1) \qquad a-a'+b+b'<c-c',$$

therefore

$$a'-a-b-b'>c'-c.$$

But by hypothesis, b, b' are positive, therefore

$$a'-a-(b+b')+2(b+b')>c'-c$$

or

$$a'-a+b+b'>c'-c,$$

and from (1) we know at once that the relations $a=a'$, $b=b'$, $c=c'$ cannot hold good at the same time since b, b' are positive. Therefore by the definition of inequality, B is greater than A.

(c). The class does not satisfy Postulate V. In this class, there are elements which do not satisfy Postulate V. For example, taking any two real numbers a, c and a positive real number b, select a', b', c'; a'', b'', c'', so that the following relations hold good,

$$(a) \qquad a'=a-k, \quad b'=b, \quad c'=c-l,$$

$$(b) \qquad a''=a+m, \quad b''=b, \quad c''=c-n,$$

where k, l, m, n satisfy the two conditions

$$(c) \qquad k+m+n-2b<l<k+2b, \quad k\neq 0,$$

$$(d) \qquad 2b<m+n, \qquad l\neq n.$$

It is evident that there are k, l, m, n satisfying the above conditions, because it is sufficient for the purpose to take m, n, so that they satisfy the relations

$$2b < m+n < 4b.$$

Now with these numbers construct the three elements $A(a, b, c)$, $B(a', b', c')$, $C(a'', b'', c'')$ of the class, then they do not satisfy Postulate V.

In the first place, A is greater than B. For, from (a) we have

$$a-a'+b+b'=a-(a-k)+b+b=k+2b$$

and

$$c-c'=c-(c-l)=l.$$

But by (c)

$$k+2b>l,$$

therefore

$$a-a'+b+b'>c-c',$$

and moreover $a \neq a'$ since $k \neq 0$. Therefore, by definition, A is greater than B.

Next B is greater than C. For, from (a) and (b) we have

$$a'-a''+b'+b''=a-k-(a+m)+b+b=-k-m+2b$$

and

$$c'-c''=c-l-(c-n)=n-l.$$

But from (c)

$$k+m+n-2b<l,$$

or

$$n-l<2b-k-m.$$

Therefore

$$a'-a''+b'+b''>c'-c'',$$

and moreover $c' \neq c''$ since $l \neq n$. Therefore, by definition, B is greater than C.

Lastly A is less than C. For, from (b) we have

$$a-a''+b+b''=a-(a+m)+b+b=-m+2b$$

and

$$c-c''=c-(c-n)=n.$$

But from (d)

$$2b<m+n,$$

or

$$2b-m<n.$$

Therefore

$$a-a''+b+b''<c-c''.$$

Accordingly, by definition, A is less than C. Thus A, B, C do not satisfy Postulate V.

Independence of the Postulates of the Second Set.

(A). Postulate I is independent of Postulates II, III, IV, V.

This may be proved by using the same class of numbers used for the proof of the independence of the same postulate of the first set, since the class of numbers also satisfies Postulates II, IV of the second set.

(B). Postulate II is independent of Postulates I, III, IV, V.

Consider the class of numbers used for the proof of the independence of Postulate II of the first set, and add to the class an element $N'(101, 1)$, and define the relations of its elements as in that case. Then this class satisfies all the postulates other than Postulate II.

Proof. (a). It is evident that the class satisfies Postulate I.

(b). The class does not satisfy Postulate II. For, take two elements $N(100, 1)$ and $N'(101, 1)$, then since $101-100<1+1$, N' is greater than N by definition; but also since $100-101<1+1$, N is greater than N'. Thus N is greater than N' while N' is greater than N, so they do not satisfy Postulate II.

(c). The class satisfies Postulate III. This may be proved in exactly the same manner as in the case (c) of the proof of the independence of Postulate II in the first set.

(d). The class satisfies Postulate IV. For, when $A(a, b)$ is equal to $B(a', b')$ and B is equal to $C(a'', b'')$, we have

$$a-a'=b+b',$$
$$a'-a''=b'+b'',$$

and accordingly

$$a-a''=b+b''+2b'.$$

But B cannot be N or N', for, if it were, the relation $a-100=b+1$ or the relation $a-101=b+1$ must hold good for certain values of a and b, which is clearly impossible, since a is an integer equal to or less than 101 and b is zero or unity. Therefore b' must be zero and so the relation $a-a''=b+b''$ must hold good, which shows that A is equal to C.

(e). The class satisfies Postulate V. This may be proved in a manner similar to the case (e) of the proof of the independence of Postulate II in the first set.

(C). Postulate III is independent of Postulates I, II, IV, V.

This may be proved by using the same class of numbers used for the proof of the independence of the same postulate of the first set, since the class of numbers also satisfies Postulates II, IV of the second set.

(D). Postulate IV is independent of Postulates I, II, III, V.

Consider the class of numbers used for the proof of the independence of Postulate IV of the first set, and define the relations of its elements as in that case. Then this class satisfies all the postulates other than Postulate IV.

Proof. That this class satisfies Postulates I, III, V was already proved; and that it also satisfies Postulate II may be seen at once from the definitions of "greater than" and "less than". Thus we have only to show that this class does not satisfy Postulate IV.

Taking two positive proper fractions $b>\frac{1}{10}$, $c>\frac{1}{20}$ and an integer a of the form $1+nd\,(d>3)$, select a', b', c'; a'', b'', c'', so that

$$a'=a, \qquad b'=b, \qquad c'=c-\frac{1}{20};$$

$$a''=a, \qquad b''=b-\frac{1}{10}, \qquad c''=c'.$$

Then $A\,(a, b, c)$, $B\,(a', b', c')$, $C\,(a'', b'', c'')$ are the elements of our class, yet they do not satisfy Postulate IV. For, firstly since

$$a=a', \quad b=b', \quad c\neq c',$$

A is equal to B by definition I, 2. Next since

$$a'=a'', \quad b'\neq b'', \quad b'-b''>c'-c'', \quad c'-c''=0,$$

B is equal to C by definition I, 3. Lastly since

$$a=a'', \quad b\neq b'', \quad b-b''>c-c'', \quad c-c''>0,$$

A is less than C by definition III, 8. Thus while A is equal to B and B is equal to C, A is not equal to C, so they do not satisfy Postulate IV.

(E). Postulate V is independent of Postulates I, II, III, IV.

Consider a class of all real numbers and define the relations of its elements as follows:— (i) When $A\,(a)$, $B\,(b)$ denote two elements of the class, A is said to be equal to B when, and only when, $a-b$ is an integer (positive, negative, or zero); and when $a-b$ is not an integer, (ii) if $a-b$ is positive, then A is said to be greater than B, and (iii) if $a-b$ is negative, then A is said to be less than B. According to these definitions, this class of numbers satisfies all the postulates other than Postulate V.

Proof. That this class satisfies Postulates I, II, III, IV may be easily proved from the above definitions. Thus we have only to prove that the class does not satisfy Postulate V. Take any real number a and take with it two numbers $b=a-\frac{1}{2}$ and $c=b-\frac{1}{2}$, then, since $a-b$ and $b-c$ are both positive and yet are not integers, $A(a)$ is greater than $B(b)$, and $B(b)$ is greater than $C(c)$. But since $a-c=1$, A is equal to C by definition. Therefore these three elements A, B, C do not satisfy Postulate V. We shall call this class of numbers the Class ($\mathfrak{F}$).

Independence of the Postulates of the Set of the Second Type.

That, in this set of postulates, every one of Postulates I, III, IV, V is independent of the others may be proved by exactly the same method as that of the first set of postulates, by using the same classes of things then used (if elements of certain classes are not repeated, repeat them), since the classes all satisfy Postulate II′. Thus we have only to prove that Postulate II′ is independent of the other postulates. Now this postulate may be divided into two parts,

(i) every element of the class is repeated at least once;

(ii) A is equal to at least one of its repeated elements.

That the part (i) is independent of Postulates I, III, IV, V may be proved by the following class of numbers.

Consider the class of all integers, every integer occurring only once in the class, and define the relations between its elements as follows. If $A(a)$, $B(b)$ denote two elements of the class, then A is said to be equal to B when, and only when, $a-b=1$; and A is said to be greater than B when, and only when, $a-b<1$; and A is said to be less than B when, and only when, $a-b>1$. Then this class of numbers satisfies all the postulates except the part (i) of Postulate II′. (We shall call this class of numbers the Class ($\mathfrak{G}$).)

Proof. (a). The class clearly satisfies Postulate I.

(b). The class does not satisfy part (i) of Postulate II′, since every element of the class occurs only once.

(c). The class satisfies Postulate III. For, when A is less than B, we have $a-b>1$. Therefore $b-a$ is negative and so the relation $b-a<1$ always holds good, which shows that B is greater than A.

(d). The class satisfies Postulate IV. For, when A is equal to B and B is greater than C, we have

$$a-b=1, \quad b-c<1.$$

Now by our supposition, no element of the class is repeated and the relation of an element to itself is to be considered as meaningless, therefore when we take any two elements of the class they are different and so $b-c$ cannot be zero. Accordingly from $b-c<1$, $b-c$ must be a negative integer, and so from

$$a-b=1,$$
$$b-c=-m \quad (m: \text{ a positive integer}),$$

it follows that $a-c$ is zero or negative and accordingly $a-c<1$. This shows that A is greater than C.

(e). The class satisfies Postulate V. This may be proved in a manner similar to the case (d).

That the part (ii) is independent of Postulates I, III, IV, V and the part (i) of Postulate II′ may be proved by using Class ($\mathfrak{B}$), every element of the class being repeated twice. For, Class ($\mathfrak{B}$) satisfies Postulates I, III, IV, V and part (i) of Postulate II′, as was already seen in the proof of the independence of Postulate II of the first set. But it does not satisfy part (ii) of Postulate II′, the element $N(100, 1)$ being greater than its repeated element $N(100, 1)$.

We add here the proof of the independence of certain propositions stated in section 1, which was left unproved there.

I. Proof that the latter part of Postulate I′ cannot be deduced from Postulates I, II, III, IV, V of both sets, when the conclusions of Postulates II, III, IV, V are not necessarily unique.

To prove this, it is sufficient to construct a class of things which satisfies Postulates I, II, III, IV, V, but not the latter part of Postulate I′.

(a). Case of the first set.

Consider all the integers twice repeated, and define the relations of its elements as follows:— If $A(a)$, $B(b)$ denote two elements of the class, then A is said to be equal to B when $a=b$; and A is said to be greater than B when $a \geqq b$; and A is said to be less than B when $a<b$. Then this class has the required property. (We shall call this class of numbers the Class ($\mathfrak{H}$)).

Proof. By the above definitions, any two elements of the class satisfy one of the three relations, but when we take two elements A, B whose values satisfy $a=b$, we have the two relations $A \circleddash B$ and $A \circledcirc B$ at the same time. So this class satisfies the former part, but not the latter part of Postulate I′.

That this class satisfies all the other postulates is seen at once; and here it is to be noted that, when we have the relations $A(=)B$ and $B(>)C$, the relation $A(>)C$ always holds good, but at the same time the relation $A(=)C$ does also. The same occurs when we have the relations $A(>)B$ and $B(>)C$.

(b). Case of the second set.

In the above class of integers, define the relations of its elements as follows:— An element A is said to be equal to an element B when $a \geqq b$; and A is said to be greater than B when $a>b$; and A is said to be less than B when $a<b$. Then this class has the required property. The proof of it may be given in a manner similar to the above. (We shall call this class of numbers the Class ($\mathfrak{H}'$).) In the Class ($\mathfrak{H}'$), when its elements A, B satisfy the relation $A(>)B$, they satisfy the relation $B(<)A$ uniquely. Hence we may infer that the latter part of Postulate I′ cannot be deduced from the second set of postulates, unless the uniqueness of the conclusion of a postulate other than Postulate II is admitted. Similarly, if, in a class of all integers, we define the equality of two elements A, B, such that they are in the relation $A(=)B$ when $a \leqq b$, and moreover, if we define the inequality of two elements as in the Class ($\mathfrak{H}'$), then from this class of integers we may infer that the latter part of Postulate I′ cannot be deduced from the second set of postulates, unless the uniqueness of the conclusion of a postulate other than Postulate III is admitted.

II. Proof that, if the relation of an element to itself be considered as meaningless, the latter part of Postulate I′ cannot be deduced from Postulates I, II, III, IV, V of the first set, even when the conclusions of Postulates III, IV, V are unique.

Consider the Class ($\mathfrak{G}$), and define the relations of its elements as follows:— If $A(a)$, $B(b)$ denote two elements of the class, then A is said to be equal to B when, and only when, $a-b=1$; and A is said to be greater than B when, and only when, $a-b<1$; and A is said to be less than B when $a-b>1$ and $a-b=1$. Then this class satisfies all the postulates other than the latter part of Postulate I′.

Proof. (a). By the above definitions, any two elements of the class satisfy one of the three relations, but when we take two elements A, B satisfying $a-b=1$, they satisfy the two relations $A(=)B$ and $A(<)B$ at the same time. Thus the class does not satisfy the latter part of Postulate I′.

(b). The class satisfies Postulate II. For, when any element $B(b)$

is given, we can always find an integer a, such that $a-b=1$. Therefore any element B has an element A equal to it.

(c). The class satisfies Postulate III. For, when A is less than B, we have

$$a-b>1 \qquad \text{or} \qquad a-b=1.$$

Therefore in both cases $a-b$ is positive, so $b-a$ is negative and therefore always

$$b-a<1.$$

Thus in this case we have the relation $B(\geqq)A$ *uniquely.*

(d). The class satisfies Postulate IV. For, when A is equal to B and, B is greater than C, we have

$$a-b=1, \quad \text{and} \quad b-c<1.$$

But, as, in this case, the relation of an element to itself is considered as meaningless, B is different from C, and accordingly we have

$$b-c=-m \qquad m>0.$$

Therefore

$$a-c=1-m<1.$$

Thus in this case we have the relation $A(\geqq)C$ *uniquely.*

Similarly it may be proved that the class satisfies Postulate V *uniquely*.

III. Proof that, if the relation of an element to itself be considered as meaningless, the proposition "if $A(\equiv)B$, then $B(\equiv)A$" cannot be deduced from Postulates I, II, III, IV, V of the first set.

Consider the Class ($\mathfrak{C}$), then it satisfies Postulates I, III, IV, V as was already proved, and moreover it is at once seen that it also satisfies Postulates II. But in this class the proposition "if $A(\equiv)B$, then $B(\equiv)A$" does not hold good, for, when $A(\equiv)B$, we have

$$a-b=1,$$

therefore

$$b-a=-1<1,$$

and so by definition we must have the relation $B(\geqq)A$.

IV. Proof that Postulate II′ cannot be deduced from Postulates II and VI.

Consider the Class ($\mathfrak{G}$) and suppose that every element of the class is repeated any number of times, then this class satisfies Postulate II and VI, but not Postulate II′. For in this class every element is repeated and has another element B, such that $B(\equiv)A$; but the element A is not in the relation $A(\equiv)A$ with its repeated element.

Independence of Postulates of the Set of the Third Type and that of the Fourth Type.

The independence of them may be proved by using the Classes ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$), ($\mathfrak{D}$), ($\mathfrak{E}$), ($\mathfrak{G}$), ($\mathfrak{H}$), and by proceeding similarly as in the cases in which they were used to prove the independence of postulates then considered.

SECTION 3. CONSISTENCY OF THE POSTULATES.

To prove that there arises no contradiction in the set of our postulates and all propositions deduced from them, it is sufficient to find a class of things which satisfies our Postulates I, II, III, IV, V and which is known as non-contradictory. For, if the set of the above postulates and propositions is not consistent, then there must necessarily arise a contradiction in any class of things satisfying these postulates.

Now such a class of things can be very easily found. For example, take a class of positive integers, every element being repeated any number of times, and define the relations of equality and inequality as usual, then we know that this class satisfies all the Postulates I, II, III, IV, V given in section 1. Thus the consistency of our postulates is established. But if we do not admit as established that the class of integers is non-contradictory, we must have another class of things which is known to be consistent, and which has the required property. To satisfy this requirement, hereafter we shall give other concrete systems of things having the required property.

SECTION 4. SUFFICIENCY OF THE POSTULATES.

Sufficiency of the Postulates of the First Set.

Postulates I, II, III, IV, V of the first set are sufficient to establish a system of propositions concerning equality and inequality. Considering any class of things satisfying the above five postulates, we shall here prove that in this class all the propositions concerning equality and inequality hold good.

Proposition I. Any element A of the class is equal to itself[1].

---

[1] In any set of postulates, that A is equal to itself must be taken as a Postulate or be proved from other postulates. For, if we define the equality of two elements $A(a, b)$, $B(a', b')$ of a class of numbers, so that they are equal when, and only when, $a+a'=b+b'$, then A is not equal to itself.

Proof. First suppose that, if possible, the relation $A \circledgt A$ holds good, then by Postulate II, there is an element B satisfying the relation $B \circleddash A$. If we take this element B, then the relations

$$B \ominus A, \qquad A \circledgt A$$

would become simultaneously consistent, and accordingly, from Postulate IV, the relation $B \circledgt A$ would follow uniquely, which contradicts the fact $B \ominus A$.

Next suppose that, if possible, the relation $A \circledless A$ holds good, then by Postulate III, the relation $A \circledgt A$ would do so also uniquely([1]), contrary to the hypothesis $A \circledless A$. Thus the two relations $A \circledgt A$, $A \circledless A$ cannot both hold good, but, by Postulate I, at least one of the three relations must hold, therefore we have the relation $A \ominus A$ uniquely.

Proposition II. If $A \ominus B$, then $B \ominus A$.

Proof. Suppose that, if possible, the relation $B \circledless A$ holds good, then by Postulate III, we should have the relation $A \circledgt B$ uniquely, contrary to the hypothesis $A \ominus B$. Next suppose that, if possible, the relation $B \circledgt A$ holds good, then from the relations

$$A \ominus B, \quad B \circledgt A,$$

we should have the relation $A \circledgt A$ uniquely by Postulate IV, contrary to Proposition I. Thus the relations $B \circledless A$, $B \circledgt A$ cannot both hold good, and therefore, by Postulate I, we have the relation $B \ominus A$ uniquely.

Proposition III. Any elements A, B of the class satisfy only one of the three relations $A \ominus B$, $A \circledgt B$, $A \circledless B$.

Proof. First, suppose that, if possible, the two relations $A \ominus B$ and $A \circledgt B$ hold good simultaneously, then, by Proposition II, from the relation $A \ominus B$, the relation $B \ominus A$ would follow uniquely; and by Postulate IV, from the relations $B \ominus A$ and $A \circledgt B$, the relation $B \circledgt B$ would follow uniquely, contrary to Proposition I.

Secondly, suppose that, if possible, the relations $A \ominus B$ and $A \circledless B$ hold good simultaneously, then, by Postulate III, from the relation $A \circledless B$, the relation $B \circledgt A$ would follow and by Postulate IV, from the relations $A \ominus B$ and $B \circledgt A$, the relation $A \circledgt A$ would follow uniquely, contrary to Proposition I.

Thirdly, suppose that, if possible, the relations $A \circledgt B$ and $A \circledless B$ hold good simultaneously, then, by Postulate III, from the relation $A \circledless B$,

---

([1]) That, in the above, the relation $A \circledless A$ cannot hold good may be proved independently of the uniqueness of the conclusion of Postulate III, by proceeding as in the first case, after having deduced the relation $A \circledgt A$ from the relation $A \circledless A$ by Postulate III.

the relation $B \circledgtr A$ would follow uniquely; and by Postulate V, from the relations $A \circledgtr B$ and $B \circledgtr A$, the relation $A \circledgtr A$ would follow uniquely, contrary to Proposition I(1). Thus no two relations can hold good simultaneously, and so, of course, the three relations cannot hold simultaneously.

Proposition IV. If $A \circleddash B$ and $B \circleddash C$, then $A \circleddash C$.

Proof. First, suppose that, if possible, the relation $A \circledgtr C$ holds good, then by Proposition II, from the relation $A \circleddash B$, the relation $B \circleddash A$ would follow; and by Postulate IV, from the relations $B \circleddash A$ and $A \circledgtr C$, the relation $B \circledgtr C$ would follow uniquely, contrary to the hypothesis $B \circleddash C$.

Next, suppose that, if possible, the relation $A \circledless C$ holds good, then by Postulate III, from the relation $A \circledless C$ the relation $C \circledgtr A$ would follow, and from the relations $B \circleddash C$ and $C \circledgtr A$, the relation $B \circledgtr A$ would follow uniquely. But, on the other hand, from the relation $A \circleddash B$ follows the relation $B \circleddash A$ uniquely, contrary to the above result.

Therefore the two relations $A \circledgtr C$ and $A \circledless C$ cannot both hold good, and so by Postulate I, we have the relation $A \circleddash C$ uniquely.

Proposition V. If $A \circledgtr B$, then $B \circledless A$.

Proof. First, suppose that, if possible, the relation $B \circleddash A$ holds good, then by Proposition II, from it the relation $A \circleddash B$ would follow uniquely, contrary to the hypothesis $A \circledgtr B$. Next, suppose that, if possible, the relation $B \circledgtr A$ holds good, then from the relations $A \circledgtr B$ and $B \circledgtr A$, the relation $A \circledgtr A$ would follow, contrary to Proposition I. Thus the two relations $B \circleddash A$ and $B \circledgtr A$ cannot both hold good, and so we have the relation $B \circledless A$ uniquely.

The above propositions I, II, III, IV, V are usually treated as axioms, therefore their proofs are given in detail. But the following being easily capable of proof in the usual manner, details are omitted.

Proposition VI. If $A \circleddash B$ and $B \circledless C$, then $A \circledless C$.

Proposition VII. If $A \circledless B$ and $B \circleddash C$, then $A \circledless C$.

---

(1) The third case may be proved independently of Postulate V. For, by Propositon I, we have the relation $A \circleddash A$, and, by Postulate IV, we have the relation $A \circledgtr B$ uniquely from the relations $A \circleddash A$ and $A \circledgtr B$. Therefore when A, B satisfy the relation $A \circledgtr B$, they satisfy it uniquely. So the relations $A \circledgtr B$ and $A \circledless B$ cannot co-exist. This method of proving Proposition III is useful when we discuss a set of postulates not containing Postulate V. Further, since, when A, B satisfy the relation $A \circledgtr B$, they satisfy it uniquely, the relations $A \circledgtr B$ and $A \circleddash B$ cannot co-exist. So we can prove Proposition III by using the uniqueness of the conclusion of only one Postulate IV.

Proposition VIII. If $A \textcircled{>} B$ and $B \textcircled{=} C$, then $A \textcircled{>} C$.

Proposition IX. If $A \textcircled{<} B$ and $B \textcircled{<} C$, then $A \textcircled{<} C$.

The five postulates and the above nine propositions are fundamental ones concerning the three relations $\textcircled{=}$, $\textcircled{>}$, $\textcircled{<}$. From them we get at once the following, by combining one or two of Postulate III and Propositions II, V to each of Postulates IV, V and Propositions IV, VI, VII, VIII, IX.

Proposition X. If $B \textcircled{=} A$ and $B \overset{\textcircled{=}}{\underset{\textcircled{<}}{\textcircled{>}}} C$, then $A \overset{\textcircled{=}}{\underset{\textcircled{<}}{\textcircled{>}}} C$.

Proposition XI. If $B \textcircled{>} A$ and $B \underset{\textcircled{<}}{\textcircled{=}} C$, then $A \textcircled{<} C$.

Proposition XII. If $B \textcircled{<} A$ and $B \underset{\textcircled{>}}{\textcircled{=}} C$, then $A \textcircled{>} C$.

Proposition XIII. If $A \textcircled{=} B$ and $C \overset{\textcircled{=}}{\underset{\textcircled{<}}{\textcircled{>}}} B$, then $A \overset{\textcircled{=}}{\underset{\textcircled{>}}{\textcircled{<}}} C$.

Proposition XIV. If $A \textcircled{>} B$ and $C \underset{\textcircled{<}}{\textcircled{=}} B$, then $A \textcircled{>} C$.

Proposition XV. If $A \textcircled{<} B$ and $C \underset{\textcircled{>}}{\textcircled{=}} B$, then $A \textcircled{<} C$.

Proposition XVI. If $B \textcircled{=} A$ and $C \overset{\textcircled{=}}{\underset{\textcircled{<}}{\textcircled{>}}} B$, then $A \overset{\textcircled{=}}{\underset{\textcircled{>}}{\textcircled{<}}} C$.

Proposition XVII. If $B \textcircled{>} A$ and $C \underset{\textcircled{>}}{\textcircled{=}} B$, then $A \textcircled{<} C$.

Proposition XVIII. If $B \textcircled{<} A$ and $C \underset{\textcircled{<}}{\textcircled{=}} B$, then $A \textcircled{>} C$.

By the above propositions, the relation of an element to itself is uniquely determined, and also when a relation of A to B is given the relation of B to A is uniquely determined. Moreover, as far as the above propositions are concerned, when the relations of A to B and B to C are given, the relation of A to C is also uniquely determined. But, of all possible cases concerning the relations of A to B and B to C, two, and only two cases

(i) $A \textcircled{>} B$ and $B \textcircled{<} C$,

(ii) $A \textcircled{<} B$ and $B \textcircled{>} C$

remain not considered above. Now let us deal with them.

In the case (i), if a class of things contains at least three elements which are not equal to one another, in other words, if a class of things satisfies non-vacuously all our five postulates[1], then whatever the class of things may be, from the above hypothesis only, the relation of A to

---

(1) In order that Postulate V may be satisfied non-vacuously it is necessary that the class should contain at least three elements which are not equal to one another.

C cannot be uniquely determined, since, in any of the above classes, from the relations $A \textcircled{>} B$ and $B \textcircled{<} C$, every one of the three relations $A \textcircled{=} C$, $A \textcircled{>} C$, $A \textcircled{<} C$ may follow. To prove it, take any three elements a_1, a_2, a_3 which are not equal to one another, and suppose that they are in the relations

$$a_1 \textcircled{>} a_2 \text{ and } a_2 \textcircled{>} a_3 \text{(1)}.$$

Then, since in the class our five postulates hold good, we have the relations

$$a_2 \textcircled{<} a_1, \quad a_3 \textcircled{<} a_2, \quad a_1 \textcircled{>} a_3 \quad \text{and} \quad a_3 \textcircled{<} a_1.$$

Therefore if we denote a_2, a_3, a_1 by A, B, C respectively, then we have the three relations

$$A \textcircled{>} B, \ B \textcircled{<} C \text{ and } A \textcircled{<} C$$

at the same time; thus in this class, when the relations $A \textcircled{>} B$ and $B \textcircled{<} C$ hold good, the relation $A \textcircled{<} C$ occurs.

Next if we denote a_1, a_3, a_2 by A, B, C respectively, then we have the three relations

$$A \textcircled{>} B, \ B \textcircled{<} C \text{ and } A \textcircled{>} C$$

at the same time; thus in this class, when the relations $A \textcircled{>} B$ and $B \textcircled{<} C$ hold good, the relation $A \textcircled{>} C$ occurs.

Lastly, by Postulate II, the element a_1 has an element, which is equal to it; if we denote this element by a_4, we have the relation $a_4 \textcircled{=} a_1$; and from the relations $a_4 \textcircled{=} a_1$ and $a_1 \textcircled{>} a_2$, we have the relation $a_4 \textcircled{>} a_2$ by Postulate IV. Therefore if we denote a_4, a_2, a_1 by A, B, C respectively, we have the three relations

$$A \textcircled{>} B, \ B \textcircled{<} C \text{ and } A \textcircled{=} C$$

at the same time; thus in this class when the relations $A \textcircled{>} B$ and $B \textcircled{<} C$ hold good, the relation $A \textcircled{=} C$ occurs.

Thus from the relations $A \textcircled{>} B$ and $B \textcircled{<} C$, every one of the three relations $A \textcircled{=} C$, $A \textcircled{>} C$, $A \textcircled{<} C$ occurs in any class of things satisfying our five postulates non-vacuously.

In the case (ii) a similar consideration leads to the same result. Thus we may conclude:

In any class of things satisfying our five postulates non-vacuously

---

(1) We may suppose that they are always in these relations without the loss of generality, since any three elements satisfying our postulates always may be put in these relations when they are not equal to one another.

the relation of A to C cannot be determined uniquely from the relations $A \circledgtr B$ and $B \circledless C$, or the relations $A \circledless B$ and $B \circledgtr C$.

From the above, it follows at once that, in any class of things satisfying our five postulates, when any n elements A_1, A_2, $A_3, \cdots\cdots$, A_n are taken and the relations of A_1 to A_2, A_2 to $A_3, \cdots\cdots$, A_{n-1} to A_n are given, the relation of A_1 to A_n is uniquely determined *in general*. Only in the case where one or both of the relations (i), (ii) enter in the above $(n-1)$ relations it is not determined, and this indetermination always occurs whatever the class of things may be. So we may conclude:

In a class of things satisfying our five postulates, whatever the class of things may be, the same propositions concerning equality and inequality always hold good; or, in other words, from any hypothesis whatever, provided that it holds true in the class of things considered, one and the same conclusion is always obtained, whatever the class of things may be.

Sufficiency of the Second Set of Postulates.

To prove the sufficiency of this set of postulates, it is sufficient to show that all the postulates of the first set are deduced from this set of postulates. But, since in both sets Postulates I, III, V are identical, we have only to deduce Postulates II, IV of the first set from the second. Now to deduce them we shall first prove the following propositions as lemmas.

Proposition I. A is equal to itself.

Proof. Suppose that, if possible, the relation $A \circledgtr A$ holds good, then, by Postulate II, the relation $A \circledless A$ would follow from it uniquely, contrary to the supposition $A \circledgtr A$. Similarly the supposition that the relation $A \circledless A$ may hold good leads to the same contradiction by Postulate III. Therefore by Postulate I the relation $A \circleddash A$ must hold good.

Proposition II. If $A \circleddash B$, then $B \circleddash A$.

This may be proved at once by Postulates II and III.

From these propositions we can prove that Postulates II, IV of the first set also hold good in the class of things satisfying the second set of postulates.

Postulate IV of the first set. If $A \circleddash B$ and $B \circledgtr C$, then $A \circledgtr C$.

Proof. By Proposition II, the relation $B \circleddash A$ follows from the relation $A \circleddash B$. Now, if we suppose that the relation $A \circleddash C$ holds good in this case, we should have the relation $B \circleddash C$ uniquely from the relations $B \circleddash A$ and $A \circleddash C$ by Postulate IV of the second set, contrary to the hypothesis $B \circledgtr C$.

Next, suppose that, if possible, the relation $A \circledless C$ holds good, then

by Postulate III, from it we should have the relation $C ⧁ A$ uniquely; and by Postulate V, from the relations $B ⧁ C$, $C ⧁ A$, we should have the relation $B ⧁ A$ uniquely, contrary to the relation $B ⊜ A$ which results from the hypothesis $A ⊜ B$ by Proposition II. Therefore by Postulate I we must have the relation $A ⧁ C$.

Postulate II of the first set. Any element A has an element B satisfying the relation $B ⊜ A$.

Proof. By Proposition I, any element A is equal to itself. Therefore any element has always an element, such that they satisfy the relation $B ⊜ A$, since in the relation $B ⊜ A$ of Postulate II of the first set, two elements A, B may be different from or identical with each other.

Thus from the second set of postulates, all the postulates of the first set are deduced. Conversely it was already proved that from the first set of postulates all the postulates of the second set are deduced. So both sets of postulates are identical.

Here we add a proof that Proposition III of the first set can be deduced from the uniqueness of the conclusions of only two Postulates II and III. As was shown above, Proposition II "if $A ⊜ B$, then $B ⊜ A$ uniquely" is deduced by Postulates II and III. Now by the uniqueness of the conclusions of Proposition II and Postulate II, it may easily be proved that the relations $A ⊜ B$ and $A ⧁ B$ cannot co-exist; and similarly, by Proposition II and Postulate III, that the relations $A ⊜ B$ and $A ⧀ B$ cannot co-exist; and lastly, by Postulates II and III, that the relations $A ⧁ B$ and $A ⧀ B$ cannot co-exist.

Sufficiency of the Set of Postulates of the Second Type([1]).

By Postulate II, every element A of a class has another element A, such that they satisfy the relation $A ⊜ A$, and as they are the repeated elements of one and the same element, they must be considered as having the same properties in every respect, besides being distinct. Therefore, as a characteristic property of these elements, we must conclude that one of them can replace the other in every case where it occurs. For example, when a class of numbers contains two distinct integers 3 and 3, as there is nothing to distinguish the former 3 from the latter 3, we can always replace the former 3 by the latter 3 and vice versâ, in any relation whatever. But as the relation $⊜$ is an undefined relation and

([1]) In this set of postulates the relation of an element to itself is considered as meaningless.

has no meaning in itself, from the above characteristic of two A's it does not follow that they either satisfy the relation $A \circleddash A$ or do not. Therefore the relation $A \circleddash A$ must be established as a postulate or as a deduction from a set of postulates as was already remarked. Now, by Postulate II, when there are n repeated elements A's, we know that any one of them has at least one element A, of which the relation is $A \circleddash A$, but that any two of them are in the same relation we must demonstrate.

Proposition I. Every element A is equal to its repeated element A (when there are more than two such elements).

Proof. Suppose that the class has n distinct elements, each of which is the repeated one of an element A, and denote them by A_1, $A_2, \cdots\cdots$, A_n, for the convenience of distinction, then by Postulate II, any element A_j of them has an element A_k, such that they satisfy the relation $A_k \circleddash A_j$. Now take any element A_l, different from A_j and A_k, and replace A_k by A_l, then we have $A_l \circleddash A_j$. Therefore any element A is equal to any repeated element of A.

Proposition II. If $A \circleddash B$, then $B \circleddash A$.

Proof. Suppose that, if possible, the relation $B \circledless A$ holds good, then by Postulate III, from it we should have the relation $A \circledgtr B$ uniquely, contrary to the hypothesis $A \circleddash B$. Next, suppose that, if possible, the relation $B \circledgtr A$ holds, and replace A of $A \circleddash B$ by another element A; then we have

$$A \circleddash B, \quad B \circledgtr A$$

in which the former A and the latter A are distinct. Therefore we can apply Postulate IV to the above relations and we should have the relation $A \circledgtr A$ uniquely, contrary to Proposition I. Thus by Postulate I we have the relation $B \circleddash A$ uniquely.

All the other propositions may be similarly proved by replacing an element A by another element A, whenever two elements A's entering in the given relations are one and the same element and so Postulates IV, V cannot be applied.

Sufficiency of the Postulates of the Sets of the Third and Fourth Types.

The sufficiency of the postulates of these sets may also be proved in a manner similar to the preceding cases.

Set of Independent Postulates of the Fifth Type.

We have already established the several sufficient sets of postulates

admitting that any element A may be replaced by itself or by its repeated element. But if it is desired to give no meaning whatever to the elements of a class of things or to the relations $(=)$, $(>)$, $(<)$, except those given by the postulates themselves, we must make a certain modification in the previous sets. For, if we do not admit that any element may be replaced by itself or by its repeated element, then the proposition "any element A is in the relation $A(=)A$" (the latter A may be the former A itself or its repeated element) cannot be deduced from any one of the sets of the four types, although we admit the relation of an element to itself and the uniqueness of the conclusions of the postulates.

To understand the truth of the above assertion, consider a class of all positive integers thrice repeated, and denote them, for the convenience of distinction, by $A_1, A_2, A_3, \cdots\cdots$; $A_1', A_2', A_3', \cdots\cdots$; $A_1'', A_2'', A_3'', \cdots\cdots$; $A_1''', A_2''', A_3''', \cdots\cdots$, the latter three being the repeated integers of the first. And define the relations of its elements as follows.

(i). Every element is equal to itself and to its repeated elements and vice versâ, except A_1, A_1', A_1'', which are in the relations

$$\begin{array}{llll} A_1'''(=)A_1, & A_1''(=)A_1', & A_1'(=)A_1'', & A_1(=)A_1''', \\ A_1''(>)A_1, & A_1'''(<)A_1', & A_1'''(<)A_1'', & A_1''(>)A_1''', \\ A_1'(>)A_1, & A_1(<)A_1', & A_1(<)A_1'', & A_1'(>)A_1''', \\ A_1(=)A_1, & A_1'(=)A_1', & A_1''(=)A_1'', & A_1'''(=)A_1'''. \end{array}$$

(ii). Any two different elements $A(a)$, $B(b)$ are in the relation $A(>)B$ or in the relation $A(<)B$ according as $a>b$ or $a<b$. Then this class of numbers satisfies Postulates I, I′, II, II′, III, IV, V, but the element A_1 is not in the relation $A_1(=)A_1'$ with its repeated element A_1'.

The same may be said of the second set of the first type. To prove it, consider the above class of numbers twice repeated and define the relations of its elements as follows.

(i). Every element is equal to itself and to its repeated elements and vice versâ, except A_1, A_1', A_1'', which are in the relations

$$\begin{array}{lll} A_1(<)A_1', & A_1'(>)A_1\,; & \\ A_1(<)A_1'', & A_1''(>)A_1\,; & \\ A_1'(=)A_1'', & A_1''(=)A_1'\,; & \\ A_1(=)A_1, & A_1'(=)A_1', & A_1''(=)A_1''. \end{array}$$

(ii). Any two different elements $A(a)$, $B(b)$ are in the relation $A(>)B$ or in the relation $A(<)B$ according as $a>b$ or $a<b$.

According to these definitions, this class of numbers satisfies all the

postulates of the second set, but the element A_1 is not in the relation $A_1 \circledequal A_1'$ with its repeated element A_1'.

Thus, in all cases, to have a sufficient set of postulates, we have to add at least one postulate "any element A of the class is in the relation $A \circledequal A$, the latter A being the former A itself, or any repeated element of the former" to the set of postulates already established. But, when this postulate is added to the first set of postulates, Postulate II becomes superfluous. Therefore replacing it by the above postulate we have a sufficient set of independent postulates, in which we need not give any meaning whatever to the elements of a class of things or to the relations $\circledequal$, $\circledgtr$, $\circledless$, except that given by the postulates themselves. When this set is written in full, it is as follows.

First set of postulates of the fifth type.

I. *Any two elements A, B of a class satisfy one, and only one, of the three relations $A \circledequal B$, $A \circledgtr B$, $A \circledless B$.*

II. *Every element A of the class is in the relation $A \circledequal A$, the latter A being the former A itself, or any repeated element of the former A.*

III. *If $A \circledless B$, then $B \circledgtr A$.*

IV. *If $A \circledequal B$ and $B \circledgtr C$, then $A \circledgtr C$.*

V. *If $A \circledgtr B$ and $B \circledgtr C$, then $A \circledgtr C$.*

In this set of postulates, the independence of every one of Postulates I, III, IV, V from the other postulates may be proved by using the Classes ($\mathfrak{A}$), ($\mathfrak{H}$), ($\mathfrak{C}$), ($\mathfrak{D}$), ($\mathfrak{E}$) respectively, since all of them satisfy the new Postulate II. And the independence of Postulate II has just been proved.

Further, in this set, the sufficiency of the postulates may be proved by proceeding in the same way as in the first part of this section, since, by Postulate II, the relation of an element to itself is admitted and any element is defined to be equal to itself. Moreover, when there are repeated elements $A_1, A_2, \cdots\cdots$ of an element A, they are in the relations $A \circledequal A_p$ ($p=1, 2, 3, \cdots\cdots$) by Postulate II; and that they also satisfy the relations $A_p \circledequal A$ and $A_p \circledequal A_q$ may be proved by Postulate II and Propositions II, IV, which are proved by our set of postulates (of course, in this case, the replacement of an element by its repeated element is not admitted).

Remark. When the relation of an element to itself and the replacement of an element by its repeated elements are both admitted, Postulate II "*any element A of the class has at least one element B, such that they*

satisfy the relation $B \circleddash A$" combined with Postulates I, III, IV, V of the first set form a sufficient set of postulates (sets of the first and third types). But when the relation of an element to itself is not admitted, while the replacement of an element by its repeated elements is, Postulate II is to be replaced by Postulate II′ "*any element A of the class has at least one repeated element A', such that they satisfy the relation $A' \circleddash A$*" to form a sufficient set of postulates (sets of the second and fourth types). And further when both of the above are not admitted, Postulate II′ is to be replaced by Postulate II″ "*any element A has its repeated elements such that they (A and its repeated elements) are all equal to one another*" to form a sufficient set of postulates(1). But, in the second and third cases, the proposition "any element of the class is equal to itself" has no meaning. In order that this fundamental proposition may also hold good in our class of things, it is necessary to replace Postulate II″ by Postulate II‴ "*any element A is always in the relation $A \circleddash A$, the latter A being the former A itself or any repeated element of the former A*". When Postulate II″ is replaced by Postulate II‴, the set is identical with the set of the fifth type mentioned above, and is a sufficient set of independent postulates, in which the relation of an element to itself is explicitly defined while the replacement of an element by its repeated element is not admitted(1).

---

(1) Proof of the sufficiency of the set of Postulates I, II″, III, IV, V when the relation of an element to itself and also the replacement of an element by its repeated element are not admitted.

I. If $A \circleddash B$, then $B \circleddash A$.

By Postulate II″, A has a repeated element A_1, such that they satisfy the relations $A \circleddash A_1$, and $A_1 \circleddash A$. Now suppose that, if possible, the relation $B \circledgtr A$ holds good, then A_1 and B must be in the relation $A_1 \circledless B$. For, if the relation $A_1 \circleddash B$ holds, then from the relations $A_1 \circleddash B$ and $B \circledgtr A$, the relation $A_1 \circledgtr A$ would follow by Postulate IV, contrary to the relation $A_1 \circleddash A$. Similarly the relation $A_1 \circledgtr B$ cannot hold good; so by Postulate I the relation $A_1 \circledless B$ must hold. But, by Postulate III, from this relation $A_1 \circledless B$, we must have the relation $B \circledgtr A_1$, and from the relations $A \circleddash B$ and $B \circledgtr A_1$, we should have the relation $A \circledgtr A_1$ by Postulate V, contrary to the relation $A \circleddash A_1$. Therefore the relation $B \circledgtr A$ cannot hold good.

Next, if the relation $B \circledless A$ hold good, then, by Postulate III, the relation $A \circledgtr B$ would also hold, contrary to the hypothesis $A \circleddash B$. Therefore also the relation $B \circledless A$ cannot hold good. Accordingly, by Postulate I, the relation $B \circleddash A$ holds good uniquely.

II. If $A \circledgtr B$, then $B \circledless A$.

This may be proved by exactly the same way as in the above by using Proposition I and the given postulates.

All other propositions may be proved in the usual manner by using Propositions I, II and the given postulates.

Further, when we add Postulate II$'''$ to the second set of the first type, we have another sufficient set of independent postulates.

Second set of postulates of the fifth type.

I. *Any two elements A, B of a class satisfy one, and only one, of the three relations $A \textcircled{=} B$, $A \textcircled{>} B$, $A \textcircled{<} B$.*

II. *Every element A is in the relation $A \textcircled{=} A$. (The latter A may be the former A itself, or a repeated element of the former A).*

III. *If $A \textcircled{>} B$, then $B \textcircled{<} A$.*

IV. *If $A \textcircled{<} B$, then $B \textcircled{>} A$.*

V. *If $A \textcircled{=} C$ and $B \textcircled{=} C$, then $A \textcircled{=} C$.*

VI. *If $A \textcircled{>} B$ and $B \textcircled{>} C$, then $A \textcircled{>} C$.*

Independence of the postulates.

(a). Postulate I is independent of the other postulates.

This may be proved by using the Class $(\mathfrak{A})$ and the Class $(\mathfrak{H}')$ since they also satisfy the new Postulate II.

(b). Postulate II is independent of the other postulates.

This has just been proved.

(c). Postulate III is independent of the other postulates.

Consider a class of all positive integers, every element being repeated once, and denote them by $A_1, A_2, \cdots\cdots$ and $A_1', A_2', \cdots\cdots$, and define the relations of its elements as follows.

(i). Every element is equal to itself and also to its repeated element and vice versâ. (ii). Any two different elements $A(a)$, $B(b)$ are in the relation $A \textcircled{>} B$ or in the relation $A \textcircled{<} B$ according as $a>b$ or $a<b$, except the four elements A_1, A_1', A_2, A_2', which are in the relations

$$A_1 \textcircled{=} A_2, \qquad A_2 \textcircled{=} A_1;$$
$$A_1' \textcircled{=} A_2', \qquad A_2' \textcircled{>} A_1';$$
$$A_1' \textcircled{=} A_2, \qquad A_2 \textcircled{>} A_1';$$
$$A_1 \textcircled{=} A_2', \qquad A_2' \textcircled{>} A_1;$$

Then it may be easily seen that this class of numbers satisfies all the postulates other than Postulate III, but Postulate III is not satisfied by the four elements A_1, A_1', A_2, A_2'.

(d). Postulate IV is independent of the other postulates.

This may be proved by using the Class $(\mathfrak{C})$, since the class also satisfies the new Postulates II, III, V.

(e). Postulate V is independent of the other postulates.

This may be proved by using the Class $(\mathfrak{D})$, since the class also satisfies the new Postulates II and III.

(f). Postulate VI is independent of the other postulates.

Consider the Class ($\mathfrak{F}$) and suppose that every element of the class is repeated any number of times, then this class satisfies all the postulates other than Postulate VI, but Postulate VI is not satisfied by the three elements $C(a)$, $B\left(a+\frac{1}{2}\right)$, $A(a+1)$.

Sufficiency of the postulates.

Proposition I. If $A \circleddash B$, then $B \circleddash A$.

This may be proved by Postulates I, III, IV.

Proposition II. If $A \circleddash B$ and $B \circledgt C$, then $A \circledgt C$.

This may be proved by Proposition I and Postulates I, IV, V, VI.

All other propositions may be easily deduced from our postulates and Propositions I, II.

Remark. To have a clear knowledge of the set of independent postulates in this and the preceding sections, we have considered several cases which may occur and several interpretations of postulates which they may have, and have established the corresponding sets of independent postulates of several types. But hereafter we shall treat several subjects concerning the three relations $\circleddash$, $\circledgt$, $\circledless$, taking the usual interpretation throughout our discussion and so the postulates are to be understood as such unless especially stated otherwise.

SECTION 5. SETS OF POSTULATES, BY WHICH ALL THE RELATIONS OF EQUALITY AND INEQUALITY ARE UNIQUELY DETERMINED.

We have already seen that, in any class of things satisfying our five postulates of the first set non-vacuously, the relation of A to C cannot be uniquely determined from either set of relations (i) $A \circledgt B$ and $B \circledless C$, (ii) $A \circledless B$ and $B \circledless C$. Thus naturally the question arises whether there may not exist a class of things which satisfies some of the five postulates vacuously, but never contradicts any one of them, and in which all the relations of equality and inequality are uniquely determined. The answer is given in the following

First Set of Postulates.

(The Corresponding Class of Things consisting of Two Groups).

In the first place, if there exist such a class, then by what has been shown before, it cannot contain three elements which are not equal to one another. Thus when we take any three elements of it, at least two

of them are equal, and so the class must consist either of one group only, all its elements being equal to one another, or of two groups, such that any two elements of the same group are equal to each other while any one of a group is not equal to any one of the other. In the first case, only one relation (equality) holds good in the class and we have therefore nothing to do with it. In the second case all three relations hold. We shall see whether in this class the relations of equality and inequality are uniquely determined or not.

Theorem. In the above class of things the relation of A to C is uniquely determined from the relations (i) $A ⧁ B$ and $B ⧀ C$, and also from the relations (ii) $A ⧀ B$ and $B ⧁ C$; this gives us the following propositions.

If $A ⧁ B$ and $B ⧀ C$, then $A ⊜ C$.

If $A ⧀ B$ and $B ⧁ C$, then $A ⊜ C$.

Proof. In the above two cases, since A is not equal to B, if A belong to a group $\{G\}$, then B belongs to another group $\{H\}$, and also since B is not equal to C, C must belong to $\{G\}$. Hence A and C belong to the same group and so must be in the relation $A ⊜ C$.

As the hypothesis of Postulate V may not occur in this class, omitting this postulate and supposing that all the other postulates hold good in this class, we shall see whether all the relations of equality and inequality are uniquely determined or not. Though this class is very simple, yet by the omission of Postulate V, another indetermination enters and obliges us to add a new postulate to determine all the relations uniquely. To show this, let us construct a class of things which consists of two groups having the above properties and moreover satisfies Postulates I, II, III, IV, but in which the relation of B to A is not determined from the given relation $A ⧁ B$.

Consider a class of numbers $\{A(a, b)\}$, where a denotes one of 3, 4, and b one of 0, 3, 6, 9, 12, 13, 14, 16, 17, 19; and divide these numbers into two groups, such that, in the first group $\{G\}$, the value of a in every element is always 4 and the value of b is one of the integers 0, 13, 14, 16, 17, 19, while, in the other group $\{H\}$, the value of a is always 3 and the value of b is one of the integers 0, 3, 6, 9, 12; and define the relations of its elements as follows.

If $A(a, b)$, $B(a', b')$ denote two elements of the class, then A is said to be equal to B when, and only when,

$$a=a',$$

and A is said to be greater than B when

$$a \neq a', \quad b \equiv b' \pmod{3},$$

or

$$a \neq a', \quad b > b', \quad b \not\equiv b' \pmod{3},$$

and A is said to be less than B when, and only when,

$$a \neq a', \quad b < b', \quad b \not\equiv b' \pmod{3}.$$

By these definitions, the elements of each of the above groups are equal to one another; and every element of $\{G\}$ is greater than any one of $\{H\}$; and any one of $\{H\}$ is greater than $G_0\,(4,0)$.

In this class, Postulates I, II, III(1) clearly hold good. To see that Postulate IV holds also in this class, take three elements $A\,(a, b)$, $B\,(a', b')$, $C\,(a'', b'')$, and suppose that they are in the relations $A \circeq B$ and $B \circledgt C$, then if A belong to $\{G\}$, B also belongs to $\{G\}$ and C must belong to $\{H\}$, therefore by what has just been stated A is greater than C. Next if A belong to $\{H\}$, B must also belong to $\{H\}$, and C to $\{G\}$, and since

$$b' > b'' \quad \text{or} \quad b' \equiv b'' \pmod{3},$$

b'' must be zero. Therefore we have

$$a \neq a'', \quad b \equiv b'' \pmod{3},$$

which shows that A is again greater than C. Thus Postulate IV always holds good in this class.

Now in this class take two elements $A\,(4, 13)$, $B\,(3, 9)$, then, since $4 \neq 3$ and $13 > 9$, we have the relation $A \circledgt B$; and since $3 \neq 4$ and $9 < 13$, we have the relation $B \circledless A$. Thus these elements A, B satisfy the proposition "if $A \circledgt B$, then $B \circledless A$."

Next take two elements $A\,(4, 0)$, $B\,(3, 6)$, then since $4 \neq 3$ and $0 \equiv 6 \pmod{3}$, we have the relation $A \circledgt B$; and since $3 \neq 4$ and $6 \equiv 0 \pmod{3}$, we have the relation $B \circledgt A$. Thus these elements A, B satisfy the proposition "if $A \circledgt B$, then $B \circledgt A$." Therefore in this class there are elements satisfying the two different following propositions having the same hypothesis:

(V′) if $A \circledgt B$, then $B \circledless A$;

(V″) if $A \circledgt B$, then $B \circledgt A$.

Namely in this class when A is greater than B, the relation of B to A is indeterminate. Therefore to have a class of things having all the relations of equality and inequality uniquely determined, we have to

(1) Henceforth, we shall use the first set of postulates always unless especially stated otherwise.

add a new postulate (V′) or (V″) or "if $A \circledgt B$, then $B \circledequiv A$ (V‴)" to Postulates I, II, III, IV. But there cannot exist a class of things satisfying Postulates I, II, III, IV, V″. For, by Postulate III, if $A \circledless B$, then $B \circledgt A$ uniquely, and by Postulate V″, if $B \circledgt A$, then $A \circledgt B$ uniquely, contrary to the hypothesis $A \circledless B$[1]. Next also there cannot exist a class of things satisfying Postulates I, II, III, IV, V‴. For, by Postulate V‴, if $A \circledgt B$ then $B \circledequiv A$ uniquely, and by Postulates I, II, III, IV, if $B \circledequiv A$, then $A \circledequiv B$ uniquely, contrary to the hypothesis $A \circledgt B$. So we have only one set of postulates to be imposed upon the class of things in question. Under these five Postulates I, II, III, IV, V′, it may easily be seen that, in the class of things consisting of two groups having the said property, all the relations of equality and inequality are uniquely determined. Moreover we may easily construct an actual class of things satisfying these postulates, and here we shall give an example of it.

Consider a class of things consisting of numbers $1-3n$ and $2+3n$ ($n=0, 1, 2, \cdots\cdots$), and define the relations of its elements as follows.

If $A(a)$, $B(b)$ denote two elements of the class, then A is said to be equal to B when, and only when, $a \equiv b$ (mod. 3), and A is said to be greater than or less than B according as $a>b$, $a \not\equiv b$ (mod 3), or $a<b$, $a \not\equiv b$ (mod. 3). Then this class of things may be divided into two groups, such that: the elements of the same group are equal to one another while any element of one group is not equal to any element of the other; and this class clearly satisfies all of Postulates I, II, III, IV, V′.

Thus we have found a required class of things in which all the relations of equality and inequality are uniquely determined when the class satisfies the five Postulates I, II, III, IV, V′. But, of course, it cannot be said that in any class of things satisfying the above five postulates the relations of equality and inequality are always uniquely determined. It is so only when the class of things consists of the two groups having the said property. Here we shall try to find a sufficient set of postulates by which all the relations of equality and inequality are always determined uniquely.

Set of postulates.

---

(1) Among the classes of things which consist of two groups having the said property and which satisfy Postulates I, II, III, IV, there may be one, certain elements, but not all, of which satisfy Postulate V″. We have already given an example of it. But there can never be one, all elements of which satisfy Postulate V″.

I_1. *Any two elements of the class satisfy at least one of the three relations $A \circleddash B$, $A \circledgtr B$, $A \circledless B$.*

II_1. *If $A \circledless B$, then $B \circledgtr A$.*

III_1. *If $A \circleddash B$ and $B \circledgtr C$, then $A \circledgtr C$.*

IV_1. *If $A \circledless B$ and $A \circledless C$, then $B \circleddash C$.*

V_1. *The class has an element A, such that it is equal to itself and there is no element to which A is in the relation $\circledgtr$* [1].

Theorem. In any class of things satisfying these five postulates non-vacuously, all the relations of equality and inequality are uniquely determined.

Proof. By Postulate V_1, any element B of the class is in the relation

$$(1)\quad A \circleddash B$$

or in the relation

$$(2)\quad A \circledless B$$

with respect to a definite element A. Now put all the elements satisfying the relation (1) and also A itself into a group $\{G\}$, and those satisfying the relation (2) into another group $\{H\}$, then by Postulate IV_1 all elements of $\{H\}$ are equal to one another. Moreover, as to the elements of $\{G\}$, any one of them, say G_r, has at least one element which is equal to it, namely the definite element A is in the relations $A \circleddash A$ and $A \circleddash G_r$. Thus, in this class, Postulate II of the first set of postulates holds good. So by combining this property to Postulates I_1, II_1, III_1, we may prove the following propositions in exactly the same manner as in the proof of the same propositions in section 4.

I. Any element is equal to itself.

II. If $A \circleddash B$, then $B \circleddash A$.

III. Any two elements of the class satisfy only one of the three relations. (See footnote on page 201.)

IV. If $A \circleddash B$ and $B \circleddash C$, then $A \circleddash C$.

V. If $A \circleddash B$ and $B \circledless C$, then $A \circledless C$.

From these propositions, we may prove the following fundamental properties of this class.

(1). All elements of $\{G\}$ are also equal to one another. For, take any two elements G_r, G_s from $\{G\}$, then from the relations $A \circleddash G_r$ and $A \circleddash G_s$, we have the relations $G_r \circleddash A$ and $A \circleddash G_s$ by Proposition II; and accordingly, by Proposition IV, we have the relation $G_r \circleddash G_s$.

---

[1] Namely the relation of A to any element B of the class is $A \circleddash B$ or $A \circledless B$, and never $A \circledgtr B$.

(2). Any element of $\{G\}$ is less than any element of $\{H\}$; and conversely any element of $\{H\}$ is greater than any element of $\{G\}$. For, take any element G_r from $\{G\}$, any element H_s from $\{H\}$, then we have the relations $A \circledless H_s$ and $A \circleddash G_r$ by the properties of the groups $\{G\}$, $\{H\}$, and accordingly we have the relations $G_r \circleddash A$ and $A \circledless H_s$, from which the relation $G_r \circledless H_s$ follows by Proposition V. Moreover by Postulate II_1 we have the relation $H_s \circledgtr G_r$ from the relation $G_r \circledless H_s$.

Thus we may state the fundamental properties of the elements of this class as follows.

1. *The elements of the class are divided into two groups* $\{G\}$ *and* $\{H\}$.
2. *All elements of each group are equal to one another.*
3. *All elements of the group* $\{G\}$ *are less than any one of the group* $\{H\}$.
4. *All elements of the group* $\{H\}$ *are greater than any one of the group* $\{G\}$.

Hence it follows at once that all the relations of equality and inequality are always determined uniquely, and also that all the propositions in section 4 (except those which have no meaning in this class of things) hold good in this class. For example, from the relations $A_1 \circledgtr A_2$, $A_2 \circledless A_3$, $A_3 \circleddash A_4$, $A_4 \circledless A_5$, the relation $A_1 \circledless A_5$ follows uniquely.

SECOND SET OF POSTULATES.

(The Corresponding Class of Things consisting of Three Groups)

In the foregoing pages, we have found a set of postulates, such that none of the propositions deduced from these postulates contradicts those of the usual system and by these postulates all the relations of equality and inequality are uniquely determined; and we have seen that all classes of things satisfying these postulates consist of only two groups, the elements of each group being equal to one another, and any two elements, each taken from different groups, never being equal to each other.

Now let us see, a step further, whether there may not exist a class of things which consists of three groups, such that the elements of each group are equal to one another, and any two elements, each taken from different groups, are not equal to each other, and in which all the relations of equality and inequality are uniquely determined.

In order that all propositions deduced from a new set of postulates should be coincident with those of the usual system as much as possible, we shall start with a set of postulates which are obtained only by replacing Postulate V of the usual system by the postulate "if $A \olessthan B$ and $A \olessthan C$, then $B \circledequal C$," namely with the set of the following postulates.

I. Any two elements of a class satisfy at least one of the three relations.

II. Every element A of the class has at least one element B satisfying the relation $B \circledequal A$.

III. If $A \olessthan B$, then $B \ogreaterthan A$.

IV. If $A \circledequal B$ and $B \ogreaterthan C$, then $A \ogreaterthan C$.

V. If $A \olessthan B$ and $A \olessthan C$, then $B \circledequal C$.

But in a class of things satisfying these postulates there may occur an indetermination; for, there is a class of things which satisfies these postulates, yet in which two different relations $B \olessthan A$ and $B \ogreaterthan A$ occur when A, B are in the relation $A \ogreaterthan B$.

For example, in a class of numbers -3, 12, 14, 20, 37, 67, if we define the relations of two numbers $A(a)$, $B(b)$ as follows:

1. if $a \equiv b \pmod{3}$, then $A \circledequal B$;
2. if $a \not\equiv b \pmod{3}$ and $a \equiv b \pmod{5}$, then $A \ogreaterthan B$;
3. if $a \not\equiv b \pmod{3}$, $a \not\equiv b \pmod{5}$ and $a > b$, then $A \ogreaterthan B$;
4. if $a \not\equiv b \pmod{3}$, $a \not\equiv b \pmod{5}$ and $a < b$, then $A \olessthan B$,

then *they may be divided into three groups*

$$\{F\} \qquad F_1 = 37, \qquad F_2 = 67,$$
$$\{G\} \qquad G_1 = 14, \qquad G_2 = 20,$$
$$\{H\} \qquad H_1 = -3, \qquad H = 12,$$

such that (*i*) *the elements of the same group are equal to one another; and* (*ii*) *any element of* $\{F\}$ *is greater than those of* $\{G\}$ *and conversely any element of* $\{G\}$ *is less than those of* $\{F\}$; *and* (*iii*) *any element of* $\{G\}$ *is greater than those of* $\{H\}$ *and conversely any element of* $\{H\}$ *is less than those of* $\{G\}$; *and* (*iv*) *moreover, by definition* 2, *any element of* $\{F\}$ *is greater than those of* $\{H\}$, *while any element of* $\{H\}$ *is greater than those of* $\{F\}$(1).

Here it may be easily seen that this class of things satisfies the above five postulates; but when we take two elements from $\{F\}$ and $\{G\}$, or from $\{G\}$ and $\{H\}$, we have the proposition "if $A \ogreaterthan B$, then

(1) This class of things is an example presenting very striking features concerning equality and inequality.

$B \circledless A$" holding good between them; and when we take two elements from $\{F\}$ and $\{H\}$, we have another proposition "if $A \circledgtr B$, then $B \circledgtr A$" holding good between them.

Thus to avoid this indetermination, we must add one of the following three postulates:

(VI′) if $A \circledgtr B$, then $B \circledless A$,

(VI″) if $A \circledgtr B$, then $B \circledgtr A$,

(VI‴) if $A \circledgtr B$, then $B \circleddash A$.

But there cannot exist a class of things, all of whose elements satisfy Postulates I, II, III, IV, V, VI″ or Postulates I, II, III, IV, V, VI‴[1], though there may exist many classes of things which satisfy Postulates I, II, III, IV, V, and some of whose elements satisfy Postulate VI″ while the others satisfy Postulate VI′ as was already seen in the preceding example.

Thus we have the following set of postulates as a possible one.

I_2. *Any two elements of a class satisfy at least one of the three relations* $\circleddash$, $\circledgtr$, $\circledless$.

II_2. *If* $A \circledless B$, *then* $B \circledgtr A$.

III_2. *If* $A \circledgtr B$, *then* $B \circledless A$.

IV_2. *If* $A \circleddash B$ *and* $B \circledgtr C$, *then* $A \circledgtr C$.

V_2. *If* $A \circledless B$ *and* $A \circledless C$, *then* $B \circleddash C$.

From this set of postulates the relation $A \circleddash A$ may be easily deduced; so Postulate II is always satisfied by any class of things satisfying these postulates, and accordingly it was omitted from our set of postulates. Also from the above postulates the following propositions may be easily deduced.

I. If $A \circleddash B$, then $B \circleddash A$.

II. If $A \circleddash B$ and $B \circleddash C$, then $A \circleddash C$.

III. Any two elements of the class satisfy only one of the three relations.

Now we shall inquire (I) whether there is a class of things satisfying these postulates, and if there be such, what properties it must have; and (II) whether these postulates are sufficient to have all the relations of equality and inequality uniquely determined.

Suppose that there is a class of things satisfying these postulates,

(1) This may be easily proved in exactly the same manner as in the proof of the similar propositions given on p. 214.

then any element of the class satisfies one, and only one, of the following three relations with respect to a definite element A,

$$(1)\quad A\textcircled{>}B, \qquad (2)\quad A\textcircled{=}B, \qquad (3)\quad A\textcircled{<}B.$$

Now put all the elements satisfying the relation (1) into the first group $\{F\}$, and those satisfying the relation (2) and also A itself into the second group $\{G\}$, and those satisfying the relation (3) into the third group $\{H\}$, then these groups have the following interesting properties.

Property I. All elements of any one of the three groups are equal to one another.

Proof. By Postulate V_2, all the elements of $\{H\}$ are equal to one another, and by Propositions I, II, all the elements of $\{G\}$ are equal to one another, and that all the elements of $\{F\}$ have also the same property may be proved as follows.

Take two elements F_r, F_s from $\{F\}$, then we have the relations $A\textcircled{>}F_r$, and $A\textcircled{>}F_s$. Suppose that, if possible, the relation $F_r\textcircled{>}F_s$ hold good, then by Postulate III_2 we would have the relation $F_s\textcircled{<}F_r$, and also from the relation $A\textcircled{>}F_s$, we have the relation $F_s\textcircled{<}A$ by the same postulate. Whence we would have the relation $A\textcircled{=}F_r$ uniquely by Postulate V_2, contrary to the property of F_r, since F_r satisfies the relation $A\textcircled{>}F_r$ only.

Next, if the relation $F_r\textcircled{<}F_s$ hold good, then from the relations $F_r\textcircled{<}F_s$ and $F_r\textcircled{<}A$ we should have the relation $A\textcircled{=}F_s$ by Postulate V_2, which again contradicts the property of F_s. Therefore we must have the relation $F_r\textcircled{=}F_s$.

Property II. All the elements of $\{H\}$ are greater than those of $\{G\}$, and all the elements of $\{G\}$ greater than those of $\{F\}$, and all the elements of $\{F\}$ greater than those of $\{H\}$; and conversely all the elements of $\{F\}$ are less than those of $\{G\}$, and all the elements of $\{G\}$ less than those of $\{H\}$, and all the elements of $\{H\}$ less than those of $\{F\}$.

Proof. Denote any elements of $\{G\}$ and $\{H\}$ by G_s and H_t respectively, then we have the two relations

$$A\textcircled{=}G_s, \qquad A\textcircled{<}H_t.$$

Now suppose that the relation $H_t\textcircled{=}G_s$ hold good, then from the relations $A\textcircled{=}G$ and $H_t\textcircled{=}G_s$, we should have the relation $A\textcircled{=}H_t$ uniquely (Propositions I, II), which contradicts the relation $A\textcircled{<}H_t$. Next suppose that the relation $H_t\textcircled{<}G_s$ hold good, then from it we should have the relation $G_s\textcircled{>}H_t$ (Postulate II_2), and from the relations $A\textcircled{=}G_s$ and

$G_s(\gtrdot)H_t$ we should have the relation $A(\gtrdot)H_t$ uniquely (Postulate IV_2), which again contradicts the relation $A(\lessdot)H_t$. Therefore by Postulate I_2 we have the relation $H_t(\gtrdot)G_s$.

Secondly, denote any element of $\{F\}$ by F_r, then we have the two relations

$$A(\gtrdot)F_r, \qquad A(\equiv)G_s.$$

Therefore the two relations $G_s(\equiv)A$ and $A(\gtrdot)F_r$ hold good simultaneously. Hence the relation $G_s(\gtrdot)F_r$ always holds (Postulate IV_2).

Thirdly, the relation $F_r(\gtrdot)H_t$ always holds good, for suppose that, if possible, the relation $F_r(\lessdot)H_t$ hold, then from the relations $F_r(\lessdot)H_t$ and $F_r(\lessdot)A$, we should have the relation $H_t(\equiv)A$ uniquely (Postulate V_2), which contradicts the relation $H_t(\gtrdot)A$. Next if the relation $F_r(\equiv)H_t$ hold, then from the relations $F_r(\equiv)H_t$ and $H_t(\gtrdot)A$, we should have the relation $F_r(\gtrdot)A$ uniquely (Postulate IV_2), which again contradicts the relation $F_r(\lessdot)A$. Therefore we have the relation $F_r(\gtrdot)H_t$.

Thus the former part of Property II is proved, and the latter part of it follows at once from the former part of Property II and Postulate III_2.

Property III. In any class of things satisfying the above postulates, all the relations of equality and inequality are uniquely determined.

This follows at once from Properties I and II.

The Propositions I, II, III, IV, V, VI, VII, VIII of the usual system are also true in this class of things. But here instead of Postulate V and Proposition IX of the usual system we have the following:

"if $A(\gtrdot)B$ and $B(\gtrdot)C$, then $A(\lessdot)C$,"

"if $A(\lessdot)B$ and $B(\lessdot)C$, then $A(\gtrdot)C$,"

which contradict those of the usual system.

Moreover in this class of things we have always the relation $A(\equiv)C$ whenever the relations $B(\lessdot)A$ and $B(\lessdot)C$, or the relations $B(\gtrdot)A$ and $B(\gtrdot)C$ hold good; and also we have the relation $A(\equiv)C$ whenever the relations $A(\gtrdot)B$ and $C(\gtrdot)B$ or the relations $A(\lessdot)B$ and $C(\lessdot)B$ hold; while in these cases the relation of A to C is not determined in the usual system.

Since the above class of things has very remarkable properties different from the usual, we shall give two examples of it, the one the arithmetical, and the other the geometrical.

Arithmetical example.

Take the nine numbers 9, 12, 18; 20, 35, 50; 64, 106, 127, and define the relations of their equality and inequality as follows.

1. If $A(a)$, $B(b)$ denote two numbers of the class, they are said to be in the relation $A ⊜ B$ when, and only when, $a \equiv b \pmod{3}$.

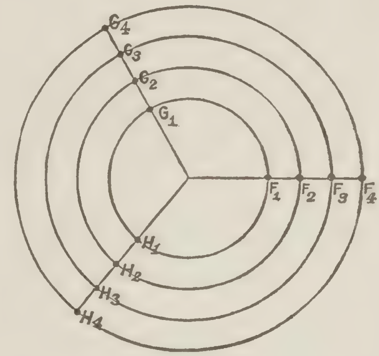
2. If $a \not\equiv b \pmod{3}$ and at least one of a, b is a multiple of 5, then they are said to be in the relation $A ⧁ B$ when $a > b$, and in the relation $A ⧀ B$ when $a < b$.

3. If $a \not\equiv b \pmod{3}$ and both of a, b are not mul iple of 5, then they are said to be in the relation $A ⧁ B$ when $R_7(a) > R_7(b)$, and in the relation $A ⧀ B$ when $R_7(a) < R_7(b)$, where $R_7(a)$ and $R_7(b)$ denote the remainders of a and b divided by 7.

By these definitions of equality and inequality, the three numbers 9, 12, 18 form the first group $\{F\}$, and the three numbers 20, 35, 50 the second group $\{G\}$, and the three numbers 64, 106, 127 the third group $\{H\}$, having Properties I, II; and this class of numbers satisfies all of Postulates I_2, II_2, III_2, IV_2, V_2.

Geometrical example.

Describe n concentric circles and draw three radii making an angle of 120° with one another, and denote the points of intersection of these radii and the circles by $F_1, F_2, \cdots\cdots, F_n$; $G_1, G_2, \cdots\cdots, G_n$; $H_1, H_2, \cdots\cdots, H_n$; and define the three relations ⧁, ⊜, ⧀ as follows.



1. When, and only when, two points are on the same radius, they are said to be in the relation ⊜ with each other.

2. When two points (say, F_r, G_s) are not on the same radius, supposing that the point F_r moves along the shorter arc $\widehat{F_r G_r}$ and the straight line $\overline{G_r G_s}$ toward G_s, if F_r move counterclockwise along the circular arc, then F_r is said to be in the relation $F_r ⧀ G_s$ with respect to G_s; and if F_r move clockwise, then F_r is said to be in the relation $F_r ⧁ G_s$ with respect to G_s.

By these definitions the points $F_1, F_2, \cdots\cdots, F_n$ form the first group $\{F\}$, and $G_1, G_2, \cdots\cdots, G_n$ the second group $\{G\}$, and $H_1, H_2, \cdots\cdots, H_n$ the third group $\{H\}$, having Properties I, II; and this class of points satisfies all of Postulates I_2, II_2, III_2, IV_2, V_2.

Now a question naturally arises whether there may exist a class of things in which all the relations of equality and inequality are uniquely determined, and which consists of more than three groups having Properties I, II stated in the preceding article. The answer is in the negative. For if there exist such a class of things, at least the three postulates I_2, II_2, III_2 must hold good in it, since the class must have Properties I, II; and accordingly neither of the two propositions

1. if $A \circledless B$ and $A \circledless C$, then $B \circledless C$;
2. if $A \circledless B$ and $A \circledless C$, then $B \circledgtr C$

can hold good universally in it[1]. Therefore in order that all the relations of equality and inequality may be uniquely determined, the proposition

(a) if $A \circledless B$ and $A \circledless C$, then $B \circleddash C$

must hold good, and similarly also the propositions

(b) if $A \circledgtr B$ and $A \circledgtr C$, then $B \circleddash C$,

(c) if $A \circleddash B$ and $A \circleddash C$, then $B \circleddash C$

must hold. From these propositions (a), (b), (c), it follows that the elements of the class can be divided into three groups, such that the elements of the same group are equal to one another. Thus they cannot be divided into more than three groups, such that the elements of the same group are equal to one another and the elements belonging to the different groups are never equal to one another.

SECTION 6. CONCRETE SYSTEMS OF THINGS FOR THE INDEPENDENCE AND CONSISTENCY PROOFS.

The set of postulates hitherto discussed is a set of conditions imposed upon the undefined terms "equal to," "greater than" and "less than," and may be considered as definitions of these terms. The science which treats of the propositions deduced from these postulates is therefore purely abstract, but admits of several concrete interpretations. We shall give some examples of it.

Consider a class of men consisting of

(1) If Proposition (1) be always true for any element of the class, then from the relations $A \circledless B$ and $A \circledless C$, the relation $B \circledless C$ would follow uniquely; and, on the other hand, from the relations $A \circledless C$ and $A \circledless B$ the relation $C \circledless B$ would follow uniquely, and accordingly by Postulate II_2, the relation $B \circledgtr C$ would follow uniquely. But in both cases the hypotheses are identical with each other, therefore the above results are absurd. The same may be said of Proposition (2).

$$
\begin{array}{l}
A_1,\ A_2,\ A_3,\cdots\cdots\cdots\cdots,\ A_{m_1},\\
B_1,\ B_2,\ B_3,\cdots\cdots\cdots\cdots,\ B_{m_2},\\
C_1,\ C_2,\ C_3,\cdots\cdots\cdots\cdots,\ C_{m_3},\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{array}
$$

where A_1, A_2, $A_3,\cdots\cdots$, A_{m_1} represent brothers of a family and B_1, B_2, $B_3,\cdots\cdots$, B_{m_2} are all the sons of the A's, C_1, C_2, $C_3,\cdots\cdots$, C_{m_3} those of the B's and so on; and we assume that m_1, m_2, $m_3,\cdots\cdots$ are each greater than two. Of this great family, taking the meaning of the words "ancestor" "descendant" and "brother" in the wider sense, we say that the A's are in the ancestral line of all the B's, C's,$\cdots\cdots$, or briefly they are the ancestors of all the B's, C's,$\cdots\cdots$; the B's are those of all the C's, D's,$\cdots\cdots$ and so on. Conversely the B's, C's, $\cdots\cdots$, are all descendants of the A's; the C's, D's$\cdots\cdots$ those of the B's and so on. Lastly we say that each of the B's is in a brotherly relation to any other one of the B's, and similarly for the C's D's,$\cdots\cdots$. Thus defining the words "ancestor," "descendant" and "brother" we proceed to interpret the undefined terms as follows.

Under the term "is greater than" we mean "is an ancestor of" and under the term "is less than" "is a descendant of" and under the term "is equal to" "is a brother of." The five postulates read then:—

I. When P and Q are any two persons of the above family, P is an ancestor or descendant or brother of Q.

II. Any person P of the above family has always a brother.

III. If P is a descendant of Q, then Q is an ancestor of P.

IV. If P is a brother of Q and Q an ancestor of R, then P is an ancestor of R.

V. If P is an ancestor of Q and Q an ancestor of R, then P is an ancestor of R.

Now it is evident that these five propositions are all true under our definition of the meaning of ancestor, descendant and brother. Thus we see that our class of men satisfies all five postulates of the first set, and also since the proposition "if P is an ancestor of Q, then Q is a descendant of P" and the proposition "if P is a brother of Q and Q a brother of R, then P is a brother of R" are both true, it satisfies all postulates of the second set. Therefore all the propositions except I, which is meaningless in this case, deduced from them in section 4 must be true for this class of men.

In the above class of men, Proposition I is meaningless; if a class of things is required in which all the postulates and all the propositions are true, then the following is an example.

Consider a system of particles moving in parallel straight lines and in the same direction, and suppose that in every line there are at least three particles; we may define the three relations as follows.

If two particles A, B are in the same line, then A is said to be equal to B; and if they are in different lines and A's line is situated on the left of B's, then A is said to be greater than B, and if A's line is situated on the right of B's, then A is said to be less than B.

According to these definitions this system of particles satisfies all the postulates of the first and second sets; and if we consider a particle A as two coincident particles moving together, then all propositions are also true in this system. We shall call this system of particles "the System $\{\mathfrak{J}\}$."

In a similar manner, many other interesting interpretations may be given to the postulates of the first and second sets.

In the independence proofs of postulates, if we use such concrete systems of things, simple ones may be obtained. For instance, in the proof of independence of Postulate IV, the number system then used was somewhat complicated. Instead of it we may here take the following simpler one.

Consider all the human beings (now living or dead), and interpret "is greater than" to mean "is an ancestor of," and "is less than" to mean "is a descendant of," and "is equal to" to mean "is in a relation other than that of ancestor or descendant." Here the words "ancestor and descendant" are to be taken in the ordinary sense, namely by descendant is meant children, children of children, and so on; and by ancestor is meant parent, parent of parent, and so on. Then the five postulates read as follows.

I. When P and Q denote any two human beings, then P is an ancestor of Q, or a descendant of Q, or neither ancestor nor descendant of Q.

II. In the class of all human beings, there is a member who is neither ancestor nor descendant of P, where P is any member of the class.

III. If P is a descendant of Q, then Q is an ancestor of P.

IV. If P is neither ancestor nor descendant of Q and Q an ancestor of R, then P is an ancestor of R.

V. If P is an ancestor of Q and Q an ancestor of R, then P is an ancestor of R.

The above propositions except the fourth are clearly true. Thus this class of human beings satisfies Postulates I, II, III, V, but not Postulate IV, of the first set. Next Postulates II and IV of the second set read as follows.

II. If P is an ancestor of Q, then Q is a descendant of P.

IV. If P is neither ancestor nor descendant of Q, and Q neither ancestor nor descendant of R, then P is neither ancestor nor descendant of R.

The former of the above two is clearly true, but the latter is not so necessarily. Thus this class of human beings satisfies Postulates I, II, III, V, but not Postulate IV of the second set(1).

We have already constructed many classes of numbers to prove the independence of our postulates. But in order to show that the independence of our postulates may be proved without using the numbers, we shall give briefly the concrete systems of things having the required properties.

Here we give the systems of things required for the proof of the independence of postulates of the first set.

(A). To the system of particles $\{\mathfrak{J}\}$, add two particles moving in the same line and in the direction opposite to that of the particles of the System $\{\mathfrak{J}\}$, and define the relations of equality and inequality of these particles as follows.

A particle A is said to be equal to a particle B when they move in the same line and in the same direction; and A is said to be greater or less than B according as A's line is on the left or on the right of B's.

According to these definitions, the system of particles clearly satisfies Postulates II, III, IV, V, but Postulate I is not satisfied since two particles moving in the same line and in opposite directions do not satisfy any of the three relations.

(B). Consider a system of particles moving in parallel straight lines, which are divided into two classes, the first consisting of all lines containing only two particles moving in the same direction, the second

(1) If we admit that a person is in a certain relation, other than ancestor or descendant, to himself (for example, a person is in the relation of identity to himself), then any person is equal to himself by the definition of equality.

consisting of all lines containing more than two particles moving in different directions, and suppose that all these lines lie in one plane. Now define the relations of equality and inequality of these particles as follows.

If two particles A, B are in different lines and A's line is situated on the left of B's, then A is said to be greater than B, and if A's line is situated on the right of B's, then A is said to be less than B. If A, B are in the same line of the second class and they move in the same direction, then they are said to be equal to each other, and if they move in opposite directions and A moves upward, then A is said to be greater than B, and if A moves downward, then A is said to be less than B. If A, B are in the same line of the first class, then A is said to be greater than B. (Of course, according to this definition, B is also greater than A.)

This system of particles then satisfies Postulates I, III, IV, V, but not Postulate II. Since the demonstrations of this case are somewhat complicated, we shall give them in detail.

(i). From the above definitions it is evident that any particle A is equal to, or greater than, or less than any particle B; so Postulate I is satisfied.

(ii). Any particle in the line of the first class has no particles to which it is equal; nor it is equal to itself. So this system of particles does not satisfy Postulate II.

(iii). If A and B are in different lines and A's line is situated on the right of B's, then B's line is on the left of A's; and if A and B move in opposite directions in the same line of the second class and A moves downward, then B moves upward. Thus when A is less than B, B is always greater than A, and so Postulate III is satisfied.

(iv). If A and B move in the same direction in the same line of the second class, and (a) B and C move in opposite directions in the same line, B moving upward, then A and C move also in opposite directions in the same line, A moving upward; so by definition A is greater than C. (b) If B and C are in different lines and B's line is situated on the left of C's, then A's line is also on the left of C's; so by definition A is also greater than C. (c) The case where B and C are in the same line of the first class cannot occur, since A and B are in the same line of the second class. Thus, in all possible cases, if A is equal to B and B is greater than C, then A is greater than C. So Postulate IV is satisfied by this system of particles.

(v). That Postulate V holds good also in this system of particles may be proved as follows.

Since A is greater than B, (a) A and B are in the same line of the first class; or (b) A and B are in the same line of the second class and A moves upward while B moves downward; or (c) A and B are in different lines and A's line is on the left of B's. Next also, since B is greater than C, (d) B and C are in the same line of the first class; or (e) B and C are in the same line of the second class and B moves upward while C moves downward; or (f) B and C are in different lines and B's line is on the left of C's. From (a), (d), it follows that A and C are in the same line of the first class; and from every one of (a), (f); (b), (f); (c), (d); (c), (e); (c), (f), it follows that A and C are in different lines and A's line is on the left of C's. All other cases (a), (e); (b), (e); (b), (d) cannot occur. Thus, in all possible cases, if A is greater than B and B is greater than C, then A is also greater than C.

(C). Consider a system of particles moving in parallel straight lines, and suppose that in every straight line there are more than two particles. If two particles in the same straight line move in the same direction, then they are said to be equal to each other; and if they move in different lines and A's line is situated on the left of B's, then A is said to be greater than B; and in all other cases A is said to be less than B.

According to these definitions, the system of particles clearly satisfies our postulates except the third. But when A and B in the same straight line move in opposite directions, they do not satisfy the third postulate. Thus the system of particles satisfies Postulates I, II, IV, V, but not Postulate III.

(D). The required system of things as a class of human beings has been already given. But if a system of particles having the assigned property is required, the following will be an example.

Consider a system of particles moving in parallel straight lines, and suppose that all particles in the same line are moving in the same direction, the number of particles in one straight line being at least two. Moreover, suppose that there are at least two different lines, such that the particles of one line are moving in the direction opposite to that of the particles of the other; and also suppose that there are at least two different lines, such that all particles of those lines are moving in the same direction. Now in this system of particles define the three relations as follows.

If two particles A and B are in the same line, then they are said to be equal to each other; and if A's line is situated on the right of B's and they are moving in the same direction, then A is said to be equal to B, but if they are moving in opposite directions, then A is said to be less than B; and if A's line is situated on the left of B's, then A is said to be greater than B.

According to these definitions, the particles of this system clearly satisfy Postulates I, II, III, V. But when we take three particles A, B and C, such that A's line is situated on the right of B's and they are moving in the same direction, and B's line is situated on the left of C's, then, by definition, A is equal to B and B is greater than C, but A is not necessarily greater than C. Thus this system of particles satisfies Postulates I, II, III, V, but not Postulate IV.

(E). In the system of particles given in (C), if we interchange the terms "greater" and "less" in the definitions of equality and inequality, then the system satisfies all the postulates except Postulate V. But if A and B move in the same line and in opposite directions and B and C move in the same line and in opposite directions, then A and C move in the same line and in the same direction. So in this case Postulate V does not hold good. Thus this system of particles satisfies Postulates I, II, III, IV, but not Postulate V.

SECTION 7. GENERAL DISCUSSION ON THE SETS OF POSTULATES.

In the previous sections, we have given two sets of postulates, whose postulates are independent of one another and sufficient to establish a branch of science concerning the three relations "equal to" "greater than" and "less than." Now we proceed to give a general discussion on the sets of postulates to find as many sets of independent postulates equivalent to the above as possible, and to study the relations existing between them. The propositions concerning the above three relations are those which determine the relation of the first element A_1 to the last element A_n when $(n-1)$ relations of A_1 to A_2, A_2 to $A_3, \cdots\cdots, A_{n-1}$ to A_n are given. But these propositions are all deduced from the set of propositions containing three or less than three elements of a class of things. Thus the set of postulates having the required properties is to be selected from the following fundamental ones.

(I). Any two elements of the class satisfy at least one of the three relations ⊜, ⧁, ⧀[1].

(II). $A ⊜ A$.

(III).
- III_a. If $A ⊜ B$, then $B ⊜ A$.
- III_b. If $A ⧁ B$, then $B ⧀ A$.
- III_c. If $A ⧀ B$, then $B ⧁ A$.

(IV).
- IV_a. If $A ⊜ B$ and $B ⊜ C$, then $A ⊜ C$.
- IV_b. If $A ⊜ B$ and $B ⧁ C$, then $A ⧁ C$.
- IV_c. If $A ⊜ B$ and $B ⧀ C$, then $A ⧀ C$.
- IV_d. If $A ⧁ B$ and $B ⊜ C$, then $A ⧁ C$.
- IV_e. If $A ⧀ B$ and $B ⊜ C$, then $A ⧀ C$.

(V).
- V_a. If $A ⧁ B$ and $B ⧁ C$, then $A ⧁ C$.
- V_b. If $A ⧀ B$ and $B ⧀ C$, then $A ⧀ C$.

We shall first investigate the relations of these groups of postulates.

Theorem 1. Any sufficient set of independent postulates must contain at least one of the postulates of Group V.

Proof. For, at least one of the postulates of Group V cannot be deduced from the set of all the postulates of the other Groups I, II, III, IV. To show this, consider a system of particles moving in parallel straight lines, and define the relations of its elements as follows:

1. If two particles A and B in the same line move in the same direction, then they are said to be in the relation ⊜ to each other.

2. (a) If two particles A and B in the same line move in opposite directions and A moves upward, or (b) if A and B lie in different lines and A's line is situated on the left of B's and moreover they move in opposite directions, or (c) if A and B lie in different lines and A's line is situated on the right of B's and moreover they move in the same direction, then A is said to be in the relation ⧁ with respect to B.

3. (a) If two particles A and B in the same line move in opposite directions and A moves downward, or (b) if A and B lie in different lines and A's line is situated on the left of B's and moreover they move in the same direction, or (c) if A and B lie in different lines and A's line is situated on the right of B's and moreover they

---

(1) All these postulates are to be taken to mean that from the given hypothesis the above conclusions follow uniquely. If it be not so taken, then Postulate I must be stated as follows.

Any two elements of the class satisfy one, and only one, of the three relations ⊜, ⧁, ⧀.

move in opposite directions, then A is said to be in the relation $\textcircled{$<$}$ with respect to B.

According to these definitions, it may be easily seen that this system of particles satisfies all the postulates of Groups I, II, III, IV. But if A and B lie in the same line α, and A moves upward while B moves downward, and if C lies in another line β lying on the right of the line α and moves upward, then by definition, A and B are in the relation $A \textcircled{$>$} B$ and B and C are in the relation $B \textcircled{$>$} C$, while A and C are in the relation $A \textcircled{$<$} C$. So the system of particles does not satisfy Postulate V_a. We shall call this system of particles "the System $\{\mathfrak{S}'\}$."

Theorem 2. Any sufficient set of independent postulates must contain at least one of the postulates of Group IV.

Proof. For, at least one of the postulates of Group IV cannot be deduced from the set of all the postulates of the other Groups I, II, III, V. To show this, consider a system of particles moving in parallel straight lines and define the relations of its elements as follows.

1. (a) If two particles A and B in the same line move in the same direction, occupying the same position, or (b) if A and B lie in the same line and move in opposite directions, then they are said to be in the relation $\textcircled{$=$}$ to each other.

2. (a) If two particles A and B in the same line move in the same direction and A moves before B, or (b) if A and B lie in different lines and A's line is situated on the left of B's, then A is said to be in the relation $\textcircled{$>$}$ with respect to B.

3. (a) If two particles A and B in the same line move in the same direction and A moves after B, or (b) if A and B lie in different lines and A's line is situated on the right of B's then A is said to be in the relation $\textcircled{$<$}$ with respect to B.

According to these definitions, it may be seen at once that this system of particles satisfies all the postulates of Groups I, II, III, V. But if A and B lie in the same line and move in opposite directions, and B and C also lie in the same line and move in opposite directions, then A and C move in the same direction in the same line. Therefore, by definition, A and B and B and C are in the relations $A \textcircled{$=$} B$ and $B \textcircled{$=$} C$, while A and C are in the relation $A \textcircled{$>$} C$ or in the relation $A \textcircled{$<$} C$. Thus Postulate IV is not satisfied by this system of particles.

Theorem 3. Any sufficient set of independent postulates must contain Postulate I.

Proof. For, Postulate I cannot be deduced from the set of all the

postulates of the other Groups II, III, IV, V. To show this, consider the Class $\{\mathfrak{A}\}$, then the class of numbers satisfies all the postulates of Groups II, III, IV, V, but not Postulate I.

Theorem 4. Any sufficient set of independent postulates, which does not contain one of the postulates of Group V, must contain at least one of those of Group III.

Proof. For, at least one of the postulates of Group III cannot be deduced from the set of all those of Groups I, II, IV and one of Group V. To show this, consider a class of numbers $\{A(a, b)\}$, where a denotes one of integers and b one of the numbers between 0 and $\frac{1}{10}$ $\left(\text{excluding } 0 \text{ and } \frac{1}{10}\right)$, and define the relations of its element as follows.

1. If $A(a, b)$, $B(a', b')$ denote two elements of the class, then A is said to be in the relation $\circleddash$ with respect to B when, and only when, $a=a'$, $b=b'$. 2. A is said to be in the relation $\circledgtr$ with respect to B when $a-a' \gtreqless b+b'$. (But by the property of a and b the equality $a-a'=b+b'$ never occurs.) 3. A is said to be in the relation $\circledless$ with respect to B when $a-a'<b+b'$ (excluding the case $a=a'$ and $b=b'$).

According to these definitions, it may be seen at once that this class of numbers satisfies all the postulates of Group I, II, IV, and also it satisfies Postulate V_a, while it does not satisfy Postulates V_b and III_c. For, when we take three elements $A\left(5, \frac{1}{20}\right)$, $B\left(5, \frac{1}{30}\right)$, $C\left(5, \frac{1}{20}\right)$, we have the relations $A \circledless B$, $B \circledless C$ and $A \circleddash C$; and moreover the relations $A \circledless B$ and $B \circledless A$ hold good simultaneously by the definitions of the relations $\circleddash$, $\circledgtr$, $\circledless$. We shall call this class of numbers "the Class $\{\mathfrak{J}'\}$".

Next, if we interchange the definitions of the relations $\circledgtr$, $\circledless$ in the above class of numbers, we have a class of numbers satisfying all the postulates of Groups I, II, IV and Postulate V_b, but which fails to satisfy at least one of those of Group III. We shall call this class of numbers "the Class $\{\mathfrak{K}\}$."

Theorem 5. Any sufficient set of independent postulates, which does not contain one of Postulates III_b, III_c, must contain Postulate II.

Proof. For, Postulate II cannot be deduced from any set lacking one of Postulates III_b, III_c. To show this, consider the Class $\{\mathfrak{B}\}$, and take the element $N'\left(100, \frac{1}{2}\right)$ instead of $N(100, 1)$ of that class,

then this class of numbers satisfies all the postulates of Groups I, IV, V and Postulates III_a, III_c. But it does not satisfy Postulate II, for, in this class the relation $N' \circledgtr N'$ holds good. We shall call this class of numbers "the Class $\{\mathfrak{B}''\}$."

Next in the Class $\{\mathfrak{B}''\}$, interchange the definitions of the relations $\circledgtr$ and $\circledless$, then this new class satisfies all the postulates of Groups I, IV, V and Postulates III_a, III_b, but not Postulate II.

Cor. Any sufficient set of independent postulates not containing Postulate II must contain both Postulates III_b and III_c.

Theorem 6. To construct a sufficient set of independent postulates at least one postulate must be taken from each of the five Groups I, II, III, IV, V, if the set does not contain one of Postulates III_b and III_c and also one of Postulates V_a and V_b.

This follows at once from the above theorems.

Theorem 7. All the postulates of Group III are deduced from the set of postulates of the other groups.

Proof. Postulate III_b may be deduced from Postulates I, II, IV_b, V_a; Postulate III_c may be deduced from Postulates I, II, IV_c, V_b; and Postulate III_a may be deduced from Postulates I, III_b, III_c.

Theorem 8. Postulate II is always deduced from any set of postulates containing both of Postulates III_b and III_c.

This may be proved at once.

Theorem 9. In any sufficient set of independent postulates, if the relations $\circledgtr$ and $\circledless$ are interchanged, another sufficient set of independent postulates is always obtained.

Proof. 1. That the postulates of the latter set thus obtained are independent of one another may be proved in exactly the same manner as in the proof of independence of the postulates of the former set, using the class of things then used and interchanging the definitions of the relations $\circledgtr$ and $\circledless$ in that class of things.

2. Next to prove that the latter set is sufficient to deduce all the propositions concerning the three relations from it, it is sufficient to show that from it all the postulates of the former set may be deduced. This may be done by adopting the same process as in the deduction of the postulates of the latter set from those of the former set, only interchanging the relations $\circledgtr$ and $\circledless$ in that process.

After having discussed the principal relations existing among the postulates of the five groups, we now proceed to construct all the possible sets of independent postulates sufficient to deduce all the proposi-

tions concerning the three relations $(=)$, $(>)$, $(<)$. By Theorem 6, when any sufficient set of postulates does not contain one of Postulates III_b, III_c and also one of Postulates V_a, V_b, it must contain at least five postulates, each taken from different groups. Moreover, by Theorem 7, when any set of postulates contains both Postulates V_a and V_b, all postulates of Group III are deduced from the set, if the set, besides the above, contains certain postulates belonging to Groups I, II, IV; and by Theorem 8, when any set of postulates contains both Postulates III_b and III_c, Postulate II is always deduced from the set. By these properties, we shall distinguish the three kinds of sets of postulates for the convenience of investigation.

I. Sets of postulates which do not contain one of Postulates III_b and III_c, nor one of Postulates V_a and V_b.

II. Sets of postulates which contain both of Postulates III_b and III_c, but neither of Postulates V_a or V_b.

III. Sets of postulates which contain both of Postulates V_a and V_b.

I. Sets of postulates which do not contain one of Postulates III_b, and III_c, nor one of Postulates V_a, V_b.

In this case, as was already seen in the above, the required set of postulates must consist of postulates of which at least one is taken from each of the five groups.

(A). In the first place, the case in which one, and only one, postulate is taken from each of the five groups will be considered.

Theorem 10. If a set of postulates contains Postulates I, II, III_a only, then it cannot form a sufficient set, whatever postulates(1) *of Groups IV, V are added to it.*

Proof. This may be proved by using the Class $\{\mathfrak{J}'\}$ and the Class $\{\mathfrak{K}\}$. For the Class $\{\mathfrak{J}'\}$ satisfies Postulates I, II. III_a, V_a and all those of Group IV, but not Postulate III_c; and the Class $\{\mathfrak{K}\}$ satisfies Postulates I, II, III_a, V_b and all those of Group IV, but not Postulate III_b.

Theorem 11. If a set of postulates contains Postulates I, II, III_c, then there are two, and only two, sufficient sets of independent postulates among them, when certain postulates of Groups IV and V are added to them.

(1) Of course, in this case, only one postulate is to be taken from each of Groups IV, V.

Proof. (a). The required sets are

(i) the set consisting of Postulates I, II, III_c, IV_b, V_a, and

(ii) the set consisting of Postulates I, II, III_c, IV_a, V_a.

For, these sets differ only by Postulates II and IV_a from the first set stated in section 1, which was already shown to be sufficient; and from each of the sets (i) and (ii) all the postulates of the first may be deduced at once([1]). Therefore sets (i) and (ii) are sufficient.

Next that the postulates of these sets are independent of one another may be proved as follows.

By Theorem 6, in the above case, at least one postulate must be taken from every one of the five groups to form a sufficient set and by the above it has been proved that Postulates I, II, III_c, IV_b, V_a form a sufficient set. Then any one of the above postulates must be independent of the others. For, suppose that, if possible, one of them, say IV_b, may be deduced from the other postulates, then since the above set is sufficient, all the postulates of the five groups and especially all those of Group IV must be deduced from the set of Postulates I, II, III_c, V_a. But, on the other hand, by Theorem 2, at least one postulate of Group IV must be taken to form a sufficient set in this case; in other words, at least one postulate of Group IV cannot be deduced from the set of Postulates I, II, III_c, V_a, contrary to the above result. Therefore Postulate IV_b is independent of the others. Similarly so for any other postulate.

(b). The sets of postulates to be considered here, other than the sets (i) and (ii), are the following.

(iii). Set consisting of Postulates I, II, III_c, IV_a, V_a.

(iv). Set consisting of Postulates I, II, III_c, IV_a, V_b.

(v). Set consisting of Postulates I, II, III_c, IV_b, V_b.

(vi). Set consisting of Postulates I, II, III_c, IV_c, V_a.

(vii). Set consisting of Postulates I, II, III_c, IV_d, V_b.

(viii). Set consisting of Postulates I, II, III_c, IV_d, V_b.

(ix). Set consisting of Postulates I, II, III_c, IV_e, V_a.

(x). Set consisting of Postulates I, II, III_c, IV_e, V_b.

In the first place, that the sets (iv), (v), (vii), (viii) and (x) are not sufficient may be proved by using the Class $\{\mathfrak{R}\}$, for, the class satisfies all

---

([1]) Set (i) is obtained by replacing Postulate II of the first set by Postulate "$A(=)A$," and any class of things satisfying Postulate $A(=)A$ always satisfies Postulate II of the first set while the converse is not necessarily true.

the above postulates, but not Postulate III_b. Secondly, that the sets (iii), (vi) and (ix) are not sufficient may be proved by using the following class of things.

Consider a system of particles moving in parallel straight lines and define the relations of its elements as follows.

1. (a) If two particles A and B in the same line move in the same direction, or (b) if they move in opposite directions and A moves upward, then A is said to be in the relation $\circleddash$ with respect to B.

2. (a) If two particles A and B in the same line move in opposite directions and A moves downward, or (b) if A and B move in different lines and A's line is situated on the left of B's, then A is said to be in the relation $⧁$ with respect to B.

3. If two particles A and B move in different lines and A's line is situated on the right of B's, then A is said to be in the relation $⧀$ with respect to B.

According to these definitions, this system of particles satisfies all the postulates of the sets (iii), (vi), and (ix). But it does not satisfy Postulate III_a, for, when we consider two particles A and B in the same line, if they move in opposite directions and A moves upward, A is in the relation $\circleddash$ with respect to B, while B is in the relation $⧁$ with respect to A. We shall call this system of particles "the System $\{\mathfrak{L}\}$"; and that which is obtained by interchanging the relations $⧁$ and $⧀$ in the above "the System $\{\mathfrak{L}'\}$".

Theorem 12. If a set of postulates contains Postulates I, II, III_b, then there are two, and only two, sufficient sets of independent postulates among them, when certain postulates of Groups IV and V are added to them.

Proof. In all the sets of postulates considered in Theorem 11, if we interchange the relations $⧁$ and $⧀$, we have all the sets to be considered in this theorem. Of these sets, those which correspond to the sufficient sets of independent postulates in Theorem 11 are also sufficient sets of independent postulates in this case (Theorem 9); and those which correspond to the non-sufficient sets of the former are also non-sufficient in this case.

Thus in the case (A) we have four, and only four, sufficient sets of independent postulates.

(B). In the above, we have studied all possible cases in which one, and only one, postulate is taken from every one of the five groups. Now we shall consider the case in which two or more postulates may

be taken from each of the five groups. But since Groups I and II contain only one postulate and moreover in case I only one of Postulates III_b and III_c and only one of Postulates V_a and V_b are to be taken, all possible cases to be considered here are the following.

(a). Sets of postulates containing only one postulate of Group III.

(i). Set consisting of Postulates I, II, III_a, V_a and two or more postulates of Group IV.

(ii). Set consisting of Postulates I, II, III_a, V_b and two or more postulates of Group IV.

(iii). Set consisting of Postulates I, II, III_b, V_a and two or more postulates of Group IV.

(iv). Set consisting of Postulates I, II, III_b, V_b and two or more postulates of Group IV.

(v). Set consisting of Postulates I, II, III_c, V_a and two or more postulates of Group IV.

(vi). Set consisting of Postulates I, II, III_c, V_b and two or more postulates of Group IV.

(b). Sets of postulates containing two postulates of Group III.

(vii). Set consisting of Postulate I, II, III_a, III_b, V_a and one or more postulates of Group IV.

(viii). Set consisting of Postulates I, II, III_a, III_b, V_b and one or more postulates of Group IV.

(ix). Set consisting of Postulates I, II, III_a, III_c, V_a and one or more postulates of Group IV.

(x). Set consisting of Postulates I, II, III_a, III_c, V_b and one or more postulates of Group IV.

Theorem 13. Any set of postulates containing two or more postulates of Group IV and only one postulate of every one of the other four groups cannot form a sufficient set of independent postulates.

Proof. Of the above sets of postulates, the sets (i), (ii), (iii), (vi) are all non-sufficient; and that they are so may be proved by using the Classes $\{\mathfrak{J}'\}$ and $\{\mathfrak{K}\}$. Further, among the sets (iv), (v), there are some which are sufficient, but all of them contain superfluous postulates and if the latter are deducted nothing remains but those we obtained in the preceding case (A).

Theorem 14. (i) Any set of postulates containing two postulates of Group III and one postulate of every one of the other groups always forms a sufficient set when two Postulates III_b and V_b or III_c and V_a enter in pair into it. (ii) Any of the above sets, when it does not con-

tain two Postulates III_b and V_b or III_c and V_a in pair, is always non-sufficient.

Proof. (1) First take two Postulates III_a and III_c of Group III, then all sets of postulates containing Postulates III_c and V_a in pair are as follows.

(i). Set consisting of Postulates I, II, III_a, III_c, V_a, IV_a.

(ii). Set consisting of Postulates I, II, III_a, III_c, V_a, IV_b.

(iii). Set consisting of Postulates I, II, III_a, III_c, V_a, IV_c.

(iv). Set consisting of Postulates I, II, III_a, III_c, V_a, IV_d.

(v). Set consisting of Postulates I, II, III_a, III_c, V_a, IV_e.

Of these, from the set (i), Postulate IV_b is at once deduced; and from each of the sets (iii) and (v), Postulate III_b and hence Postulate IV_b are deduced. These sets therefore satisfy all the postulates of the first set given in section 1, and accordingly they are sufficient ones. Next each of the sets (ii) and (iv) contains all the postulates of sufficient sets given in (A), and moreover it contains a superfluous Postulate III_a. If we omit this superfluous postulate we get the sufficient set given in (A). Thus the first part of the theorem is proved. The validity of the second part of the theorem may be seen at once by using the Class $\{\mathfrak{R}\}$.

(2) Secondly, take two Postulates III_a, III_b of Group III; then that in this case also the theorem is true is at once seen by interchanging the relations $\circledgt$ and $\circledlt$ in the above sets and by using Theorem 9.

(3) The third case, in which two Postulates III_b, III_c of Group III enter in the set of postulates, will be considered in the case II and it will be found that the theorem holds good in this case also.

Now we shall prove that the postulates of the sufficient sets (i), (iii) and (v) are independent of one another. But before doing this, we have to state the following theorems.

Theorems 15. Any sufficient set of independent postulates containing Postulates I, II, III_a, V_a and one or more of Group IV must also contain Postulate III_c.

This may be proved by using the Class $\{\mathfrak{J}'\}$.

Theorem 16. Any sufficient set of independent postulates containing Postulates I, II, III_c, V_a and one or more of Postulates IV_a, IV_c, IV must also contain Postulate III_a.

This may be proved by using the System $\{\mathfrak{L}\}$.

From these theorems and Theorems 1, 2, 3, 5, the independence of postulates of the sets (i), (iii) and (v) may be proved at once, and by

Theorem 9, the sets obtained by interchanging their relations $\circledgt$ and $\circledlt$ are also sufficient sets of independent postulates.

From Theorem 14 and Classes $\{\mathfrak{J}'\}$ and $\{\mathfrak{K}\}$, we see at once that all other sets of postulates in the case (b) are non-sufficient or non-independent. Therefore in the case (B) we have six, and only six, sufficient sets of independent postulates.

II. Sets of postulates which contain both of the Postulates III_b and III_c, but neither of the Postulate V_a or V_b.

Since Postulate II may always be deduced from any set containing Postulates III_b, and III_c, Postulate II need not be considered in this case at all.

Theorem 17. A set consisting of two Postulates III_b and III_c forms always a sufficient set of independent postulates, whenever one, and only one, postulate is taken from every one of the three Groups I, IV, V, and is added to the above set.

Proof. The set of Postulates I, III_b, III_c, IV_a, V_a and also the set obtained by interchanging the relations $\circledgt$ and $\circledlt$ from the above were already proved to be sufficient. Therefore to prove that any set of postulates stated in this theorem is sufficient we have only to show that from any one of them Postulate IV_a may be deduced. But this may be done at once. For example, to deduce Postulate IV_a from Postulates I, III_b, III_c, IV_b and V_a, assume that from the relations $A \circleddash B$ and $B \circleddash C$, the relation $A \circledgt C$ would follow; then since from Postulates III_b and III_c, Postulate III_a follows and hence from the relation $A \circleddash B$ the relation $B \circleddash A$ follows, from the relations $B \circleddash A$ and $A \circledgt C$ the relation $B \circledgt C$ would follow uniquely (Postulate IV_b), contrary to the hypothesis $B \circleddash C$. Next assume that the relation $A \circledlt C$ holds good, then from Postulate III_c, the relation $C \circledgt A$ would follow; and from the relations $B \circleddash C$ and $C \circledgt A$, the relation $B \circledgt A$ would follow (Postulate IV_b), contrary to the proposition $B \circleddash A$, which is deduced from Postulates III_b and III_c. Therefore by Postulate I, we must have the relation $A \circleddash C$ from the hypotheses $A \circleddash B$ and $B \circleddash C$, and thus Postulate IV_a is deduced from Postulates I, III_b, III_c, IV_b, V_a.

Further to show that the postulates of any set mentioned above are mutually independent we may use Theorems 1, 2, 3, 5. For, by Theorems 1, 2, 3, any sufficient set must contain at least one of the postulates of each of the three Groups I, IV and V, and, by Theorem 5, Cor., any sufficient set not containing Postulate II, must contain both Postulates III_b and III_c. Hence, by reasoning similarly as in the proof

of Theorem 11, we may prove the independence of postulates in the above set.

From the above, it follows at once that any set of postulates containing Postulates III_b and III_c is non-sufficient or non-independent, unless it is one of the above sets. Thus in this case we get ten, and only ten, new sufficient sets of independent postulates, every one consisting of five postulates.

III. Sets of postulates which contain both of the Postultaes V_a and V_b.

In the cases I and II, we have studied the sets containing only one of the Postulates V_a and V_b. Now we proceed to consider the sets containing both of them, but in the case II, we have also studied the sets containing both of Postulates V_a and V_b and both of Postulates III_b and III_c, so that we may omit this case. Therefore the sets to be considered here have only one of the Postulates III_b and III_c at most, hence each of the sets must contain at least Postulates I, II, V_a and V_b (Theorems 3, 5 and hypothesis). Here again for the convenience of investigation, we shall distinguish three cases as follows.

(A). Sets containing two postulates of Group III.

In this case all the sets contain at least Postulates I, II, III_a, III_b, V_a, V_b or Postulates I, II, III_a, III_c, V_a, V_b. But, by Theorem 2, those are not enough to form a sufficient set of postulates, and so at least one of Group IV must be added to them. Nevertheless when any one of the Group is added to them, Postulate V_a or Postulate V_b is deduced from the other postulates of the set thus formed (Theorem 14). Therefore in this case no new set of postulates having the required property is obtained.

(B). Sets containing only one postulate of Group III.

First consider the case in which the set contains Postulate III_c of Group III, then it must contain at least Postulates I, II, III_c, V_a, V_b and some postulates of Group IV. But when the set contains Postulate IV_b or IV_d, Postulate V_b may be deduced from other postulates of the set, and if Postulate V_b is omitted, then the set becomes identical with that obtained in (I). Next, when the set contains one or more than one of Postulates IV_a, IV_c, IV_e, it is a non-sufficient set as may be easily proved by using the System $\{\mathfrak{L}\}$. Thus in this case no new set having the required property is obtained. Secondly, the case in which the set of Postulates contains Postulate III_b of Group III may be treated in a manner similar to the above, and the same conclusion will be obtained.

Lastly we shall consider the case in which the set of postulates contains Postulate III_a of Group III. In this case, before giving new sets of postulates having the required property we shall state the following theorems.

Theorem 18. Any sufficient set of postulates containing only one postulate of Group IV and neither Postulate III_b nor III_c of Group III must contain all those of Group V.

Proof. The Class $\{\mathfrak{J}'\}$ satisfies Postulates I, II, III_a, V_a and all those of Group IV, but not Postulate V_b. Therefore the set consisting of Postulates I, II, III_a, V_a and all those of Group IV are non-sufficient. Similarly the Class $\{\mathfrak{K}\}$ satisfies Postulates I, II, III_a, V_b and all those of Group IV, but not Postulate V_a. Therefore the set consisting of Postulates I, II, III_a, V_b and all those of Group IV is also non-sufficient. Accordingly, in this case, the set must contain both V_a and V_b to be sufficient.

Cor. Any sufficient set of postulates containing neither Postulate III_b nor III_c must contain all those of Group V.

Theorem 19. Any sufficient set of postulates containing only one of Group IV and neither Postulate III_b nor III_c of Group III must contain Postulate III_a of Group III.

Proof. The System $\{\mathfrak{L}\}$ satisfies Postulates I, II, V_a, V_b, IV_a, IV_c, and IV_e, but not Postulate III_a; and the System $\{\mathfrak{L}'\}$ satisfies Postulates I, II, V_a, V_b, IV_b, and IV_d, but not Postulate III_a. Therefore any set containing Postulates I, II, V_a, V_b and one of Group IV cannot have Postulate III_a as its logical consequence.

After having established these theorems we may state the following

Theorem 20. The set consisting of Postulates I, II, III_a and both the postulates of Group V always forms a sufficient set of independent postulates when only one postulate of Group IV is added to it.

Proof. From the set consisting of Postulates I, II, III_a, V_a and V_b, the two Postulates III_b and III_c are deduced at once. Therefore when any one postulate of Group IV is added to it, the set forms a sufficient set always (Theorem 17). Next that these postulates are independent of one another may be proved by Theorems 2, 3, 5, 18 and 19. Thus we get five new sufficient sets of independent postulates.

(C). Sets containing none of the postulates of Group III.

In this case the set must consist of Postulates I, II, V_a, V_b and certain postulates of Group IV.

Theorem 21. A set of postulates consisting of all those of Groups I, II and V cannot form a sufficient set when only one postulate of Group IV is added to it.

Proof. That the above set cannot be sufficient when one, or more than one, of Postulates IV_a, IV_b and IV_d are added to it may be proved by using the System $\{\mathfrak{L}'\}$; and that the set cannot be sufficient when one, or more than one, of Postulates IV_a, IV_c and IV_e are added to it may be proved by using the System $\{\mathfrak{L}\}$.

Cor. 1. In order that a set of postulates consisting of all those of Groups I, II and V may be a sufficient one, at least two postulates of Group IV must be added to it.

Cor. 2. Any set of postulates consisting of all those of Groups I, II and V and one, or more than one, of the Postulates IV_a, IV_b and IV_d cannot be a sufficient one.

Cor. 3. Any set of postulates consisting of all the postulates of Groups I, II and V and one, or more than one, of Postulates IV_a, IV_c and IV_e cannot be a sufficient one.

Theorem 22. Any set of postulates consisting of all the postulates of Groups I, II and V and those of one of the four pairs of postulates (IV_b, IV_c) (IV_d, IV_e), (IV_b, IV_e), (IV_c, IV_d) always forms a sufficient set of independent postulates.

Proof. From any one of the above sets, Postulate III_a can be at once deduced, therefore by Theorem 20 the sets are all sufficient. Next that the postulates of every set are independent of one another may be proved by Theorems 3; 5; 18, Cor.; 21, Cor. 2 and Cor. 3.

In the above, all possible sets of postulates containing one or two of the postulates of Group IV has been considered. All other sets, namely those containing three or more postulates of Group IV, are non-sufficient (those containing Postulates IV_a, IV_b, IV_d, or Postulates IV_a, IV_c, IV_e) or non-independent (all other than those mentioned above). Thus in this case we get four, and only four, new sufficient sets of independent postulates.

Conclusion. In the above, we have considered all possible cases and have obtained twenty nine sufficient sets of independent postulates. Now we shall try to collect and classify them. Since Postulates IV_b and IV_d have a very similar property, and whenever one of them, entering into any sufficient set of independent postulates, is replaced by the other, a new sufficient set of independent postulates is always obtained, we shall call the second set thus obtained a derived set of the

first kind[1]. The same may be said of Postulates IV_c and IV_e. Further, since, whenever the relations ⧁ and ⧀ are interchanged in any sufficient set of independent postulates, a new sufficient set of independent postulates is always obtained, we shall call the second set thus obtained a derived set of the second kind. Thus we get the following table of the sufficient sets of independent postulates.

I. Sets containing one, and only one, postulate taken from every group of postulates.

Fundamental sets.	Derived sets of the first kind.	Derived sets of the second kind.	
(α)	(β)	(α')	(β')
I, II, III_c, IV_b, V_a.	I, II, III_c, IV_d, V_a.	I, II, III_b, IV_c, V_b.	I, II, III_b, IV_e, V_b

II. Sets containing two postulates of Group III and only one postulate of all other groups.

I, II, III_a, III_c, IV_a, V_a.		I, II, III_a, III_b, IV_a, V_b.	
I, II, III_a, III_c, IV_c, V_a.	I, II, III_a, III_c, IV_e, V_a.	I, II, III_a, III_b, IV_b, V_b.	I, II, III_a, III_b, IV_d, V_b.

III. Sets containing two postulates of Group III and only one postulate of Groups I, IV and V.

I, III_b, III_c, IV_a, V_a.		I, III_c, III_b, IV_a, V_b.	
I, III_b, III_c, IV_b, V_a,	I, III_b, III_c. IV_d, V_a.	I, III_c, III_b, IV_c, V_b.	I, III_c, III_b, IV_e, V_b.
I, III_b, III_c, IV_c, V_a.	I, III_b, III_c, IV_e, V_a.	I, III_c, III_b, IV_b, V_b.	I, III_c, III_b, IV_d, V_b.

IV. Sets containing two postulates of Group V and only one postulate of all other groups.

I, II, III_a, IV_a, V_a, V_b			
I, II, III_a, IV_b, V_a, V_b.	I, II, III_a. IV_d, V_a, V_b.	I, II, III_a, IV_c, V_b, V_a.	I, II, III_a, IV_e, V_b, V_a.

V. Sets containing two postulates of Groups IV and V and one postulate of Groups I and II.

I, II, IV_b, IV_c, V_a, V_b.	I, II, IV_d, IV_e, V_a, V_b.		
	I, II, IV_b, IV_e, V_a, V_b.		I, II, IV_c, IV_d, V_b, V_a.

(1) It is to be remarked that there is a class of things satisfying one of the Postulates IV_b and IV_d, but not the other. Some classes of things having such a property will be given in the next Part.

All the above sets consist of five or six postulates, the number of those containing five postulates being fourteen while the number of those containing six postulates is fifteen. In section 1, as representing the former, we have mentioned the two sets and stated their characteristic properties. As a representatives of the latter, we may take the set of Postulates I, II, III_a, IV_a, V_a, V_b and the set of Postulates I, II, IV_b, IV_c, V_a, V_b, both having the characteristic property that, when the relations $\textcircled{>}$ and $\textcircled{<}$ are interchanged in them, each of them remains the same as before. This stableness of the postulates presents a very striking contrast to the unstableness of those of the first set in section 1. Moreover, these sets are more symmetrical and elegant in form and nature than the first and second sets in section 1; and perhaps the set of Postulates I, II, III_a, IV_a, V_a, V_b is the most beautiful of all, the relations contained in each being all of the same kind. We shall write their postulates *in extenso* so as to enable the reader to take them all in at a glance.

Representative of the sets of five postulates.

The first set.

I. Any two elements A and B of the class satisfy at least one of the three relations $A\textcircled{=}B$, $A\textcircled{>}B$, $A\textcircled{<}B$.

II. Any element A has at least one element B, such that $B\textcircled{=}A$.

III. If $A\textcircled{<}B$ then $B\textcircled{>}A$.

IV. If $A\textcircled{=}B$ and $B\textcircled{>}C$, then $A\textcircled{>}C$.

V. If $A\textcircled{>}B$ and $B\textcircled{>}C$, then $A\textcircled{>}C$.

The second set.

I. The same as above.

II. If $A\textcircled{>}B$, then $B\textcircled{<}A$.

III. If $A\textcircled{<}B$, then $B\textcircled{>}A$.

IV. If $A\textcircled{=}B$ and $B\textcircled{=}C$, then $A\textcircled{=}C$.

V. If $A\textcircled{>}B$ and $B\textcircled{>}C$, then $A\textcircled{>}C$.

Representative of the sets of six postulates.

The third set.

I. The same as above.

II. $A\textcircled{=}A$.

III. If $A\textcircled{=}B$ and $B\textcircled{>}C$, then $A\textcircled{>}C$.

IV. If $A\textcircled{=}B$ and $B\textcircled{<}C$, then $A\textcircled{<}C$.

V. If $A\textcircled{>}B$ and $B\textcircled{>}C$, then $A\textcircled{>}C$.

VI. If $A\textcircled{<}B$ and $B\textcircled{<}C$, then $A\textcircled{<}C$.

The fourth set.

I. The same as above.
II. $A \mathrel{\textcircled{=}} A$.
III. If $A \mathrel{\textcircled{=}} B$, then $B \mathrel{\textcircled{=}} A$.
IV. If $A \mathrel{\textcircled{=}} B$ and $B \mathrel{\textcircled{=}} C$, then $A \mathrel{\textcircled{=}} C$.
V. If $A \mathrel{\textcircled{>}} B$ and $B \mathrel{\textcircled{>}} C$, then $A \mathrel{\textcircled{>}} C$.
VI. If $A \mathrel{\textcircled{<}} B$ and $B \mathrel{\textcircled{<}} C$, then $A \mathrel{\textcircled{<}} C$.

PART II.

Singular Sets of Postulates.

SECTION 8. FIRST SINGULAR SET OF POSTULATES.

In section 1, we remarked that the mutual relations of our postulates of the first set are very delicate and that even the slightest alteration of one of them may cause an essential change of the properties of the set Here we shall see how it happens.

In the first place, replace Postulate II "any element A of the class has always an element B, such that $B \mathrel{\textcircled{=}} A$" by the analogous Postulate II′ "any element A of the class has always an element B, such that $A \mathrel{\textcircled{=}} B$"; then the set of Postulates I, II′, III, IV, V is not identical with the set I, II, III, IV, V, though Postulates II and II′ are so analogous that they seem interchangeable. Indeed we can prove that Postulate II does not follow from the former set of postulates while Postulate II′ is the logical consequence of the latter. The latter part of the above was already proved in section 4, hence we have only to prove here the former part of it. For this purpose, it is sufficient to find a class of things which satisfies Postulates I, II′, III, IV, V, but not Postulate II. Now a class of things having such a property was already given in section 2 to prove the independence of Postulate II from the other postulates. There we remarked that the number system $\{A(a, b), N\}$ then used satisfies not only Postulates I, III, IV, V but also Postulate II′ while it does not satisfy Postulate II.

Thus the new set of postulates embraces two main branches, in one of which Postulate II always holds good while in the other it does not. The first branch is identical with the usual set of postulates already given. The second branch has properties quite different from those which one would expect. Here we shall give some of them.

For the convenience of discussion, we shall divide the elements of a class of things belonging to the second branch into two classes, the one

consisting of all elements, each of which has an element equal to it, and the other consisting of all elements, none of which has any other equal to it; and call the elements of the former kind "*ordinary*" and those of the latter kind "*singular*". Since, by Postulate II′, each element of the set has an element to which it is equal, an ordinary element has equal elements on both sides of it([1]) while a singular one has an equal element on only one side (right) of it.

Theorem 23. If A is a singular element of the class, then A is greater than itself.

Proof. Since any singular element cannot be equal to itself by its fundamental property, it must be greater or less than itself by Postulate I. But if it is less than itself, it would be greater than itself by Postulate III, and since the conclusion of the postulate is unique, it contradicts the hypothesis. Hence any singular element must be greater than itself.

Theorem 24. Any two singular elements cannot be equal to each other.

This follows at once from the definition of the singular element.

Theorem 25. If A is a singular element of a class of things and is equal to B, then B is greater than A, or in symbol, if $A \circledequal B$, then $B \circledgtr A$.

Proof. Since B cannot be equal to a singular element A, B must be less or greater than A. If B is less than A, then A would be greater than B by Postulate III, which contradicts the hypothesis that A is equal to B. Thus B must be greater than A.

Theorem 26. If A is a singular element of a class of things and A is equal to B and B is equal to C, then C is greater than A; or in symbol, if $A \circledequal B$ and $B \circledequal C$, then $C \circledgtr A$.

Proof. Since A is singular and is equal to B, we have the relation $B \circledgtr A$ by the previous theorem. Further, since A is equal to B, B cannot be singular, and moreover, since B is equal to C, C cannot be singular also. Therefore from the relation $B \circledequal C$, the relation $C \circledequal B$ follows, and from the relations $C \circledequal B$ and $B \circledgtr A$, the relation $C \circledgtr A$ follows by Postulate IV.

Theorem 27. If A is a singular element of a class of things and is greater than B (ordinary or singular), then B cannot be equal to A; and if A is an ordinary element and is greater than B (ordinary or singular), then B cannot be greater than A. Moreover, if A is a singular element

([1]) Any ordinary element A always has two elements M_1 and M_2, such that $M_1 \circledequal A$ and $A \circledequal M_2$.

and A is equal to B and B is equal to C, then A cannot be greater than C.

Proof. The first part of the theorem follows at once from the property of a singular element, and the second part may be proved as follows.

Suppose that the relation $B \circledgtr A$ holds good, then from the relations $A \circledgtr B$ and $B \circledgtr A$ the relation $A \circledgtr A$ would follow by Postulate V, but A, being an ordinary element, must be equal to itself. Hence the relation $B \circledgtr A$ cannot hold good in this case.

In the third part of the theorem the relation $C \circledgtr A$ holds good by the previous theorem. Therefore if the relation $A \circledgtr C$ holds good, from the relations $C \circledgtr A$ and $A \circledgtr C$, the relation $C \circledgtr C$ would follow (Postulate V). But, since C is an ordinary element (by the hypothesis $B \circleddash C$), the relation $C \circleddash C$ must hold good in this case, contrary to the above result. Therefore in this case the relation $A \circledgtr C$ cannot hold good.

Theorem 28. When B is a singular element of a class of things, if $A \circledless B$ and $B \circleddash C$, then $C \circledgtr A$.

Proof. Since B is singular, from the relation $B \circleddash C$ we have the relation $C \circledgtr B$ by Theorem 25, and from the relations $C \circledgtr B$ and $B \circledgtr A$, we have the relation $C \circledgtr A$ by Postulate V.

Theorem 29. (i). When A is ordinary and B is singular, if $A \circledless B$ and $B \circleddash C$, then $A \circledless C$. (ii). When A is singular and B is ordinary, if $A \circledgtr B$ and $B \circleddash C$, then $A \circledgtr C$.

Proof. (i). Since the relation $B \circleddash C$ holds good, C is an ordinary element, and by the hypothesis of (i) A is also an ordinary element. Therefore from the conclusion $C \circledgtr A$ of the previous theorem the relation $A \circledless C$ follows. Thus the first part of the theorem is proved.

(ii). In the second part assume that, if possible, the relation $A \circleddash C$ holds good, then, since A is singular, we should have the relation $C \circledgtr A$ from it, and from the relations $C \circledgtr A$ and $A \circledgtr B$, we have the relation $C \circledgtr B$ by Postulate V. But, on the other hand, since B and C are ordinary, we must have the relation $C \circleddash B$ from the relation $B \circleddash C$, which is contrary to the above result. Next assume that, if possible, the relation $A \circledless C$ holds good, then by Postulate III we should have the relation $C \circledgtr A$ from it. With this relation $C \circledgtr A$, by reasoning as above we arrive at the same contradiction. Hence we must have the relation $A \circledgtr C$.

Theorem 30. When A, B and C are any elements of a class of things, the following always hold good.

(*i*) *if* $A ⧀ B$ *and* $B ⧀ C$, *then* $A ⧀ C$;

(*ii*) *if* $A ⊜ B$ *and* $B ⧀ C$, *then* $A ⧀ C$.

This may be proved in the usual manner.

When one or both of A and B are singular, the conclusion of a given hypothesis often presents an indetermination, while, in the case of ordinary elements, from the same hypothesis the conclusion is uniquely determined. For example, when A is singular we have the propositions:

(i) if $A ⧁ B$, then $B ⧀ A$ or $B ⧁ A$;

(ii) if $A ⊜ B$ and $B ⊜ C$, then $A ⊜ C$ or $A ⧀ C$;

(iii) if $A ⧀ B$ and $B ⊜ C$, then $A ⊜ C$ or $A ⧀ C$.

Further when B is singular we have the propositions:

(iv) if $A ⧁ B$, then $B ⊜ A$ or $B ⧀ A$;

(v) if $A ⧁ B$ and $B ⊜ C$, then $A ⊜ C$ or $A ⧀ C$ or $A ⧁ C$.

That, in the cases (i), (ii), (iv), a relation other than those mentioned above cannot occur was already proved (Theorem 27), and in (iii), that this is also the case may be easily proved.

Next to see that the above indeterminations really occur, take the Class ($\mathfrak{B}$) and add to it the five number $N_2(150, 1)$, $N_1(120, 1)$, $A_2(149, 0)$, $A_1(119, 0)$ and $A_0(100, 0)$ (any element of the class being repeated any number of times), then this new class satisfies all of the Postulates I, II′, III, IV, V. In this new class, $N(100, 1)$, N_1 and N_2 are singular while all the other elements are ordinary. Now, in this class, according to the definitions of the three relations ⊜, ⧁, ⧀, we have the following relations.

(i). The relations $N ⧁ A_0$ and $A_0 ⧁ N$, and also the relations $N ⧁ N_2$ and $N_2 ⧀ N$ hold good simultaneously.

(ii). The relations $N ⊜ A(99, 0)$, $A ⊜ A$ and $N ⊜ A$ hold good simultaneously.

(iii). The relations $N_2 ⧀ N$, $N ⊜ A$ and $N_2 ⧀ A$ hold good simultaneously.

(iv). The relations $A ⧁ N$ and $N ⊜ A$, and also the relations $A'(90, 0) ⧁ N$ and $N ⧀ A'$ hold good simultaneously.

(v). The relations $A ⧁ N$, $N ⊜ A$ and $A ⊜ A$; and the relations $A' ⧁ N$, $N ⊜ A$ and $A' ⧁ A$; and also the relations $A_0 ⧁ N$, $N ⊜ A$ and $A_0 ⧀ A$ hold good simultaneously. Further to see that two sets of relations (ii) $A ⊜ B$, $B ⊜ C$, $A ⧀ C$, and (iii) $A ⧀ B$, $B ⊜ C$, $A ⊜ C$ may occur in a class of things satisfying Postulates I, II′, III, IV, V, let us take the three numbers 1, 2, 4 (any element being repeated any number of times) and define their relations as follows.

$$
\begin{array}{lll}
1 ⊜ 1, & 2 ⊜ 2, & 4 ⧁ 4, \\
1 ⊜ 2, & 2 ⊜ 1, & 4 ⧀ 2, \\
1 ⧁ 4, & 2 ⧁ 4, & 4 ⊜ 1.
\end{array}
$$

Then they clearly satisfy Postulates I, II′, III, IV, V, and the two elements 1 and 2 are ordinary while 4 is singular. In this class of numbers the following relations hold good simultaneously.

(ii) $\qquad 4 ⊜ 1, \quad 1 ⊜ 2, \quad 4 ⧀ 2$;

(iii) $\qquad 4 ⧀ 2, \quad 2 ⊜ 1, \quad 4 ⊜ 1$.

Thus from these two classes of numbers we see that all the indeterminations (i), (ii), (iii), (iv), (v) may really occur.

Three sets of postulates.

From the set of postulates I, II′, III, IV, V, we have derived two sets of postulates, one of which satisfies Postulate II, while the other does not; and also we have seen that the properties of these two sets are very different from each other. If we interchange the relations ⧁ and ⧀ in the above first and second sets we get the third and fourth sets corresponding to them. But the first and third sets are identical with each other in substance though they are different in form, since from one of them the other is deduced. Thus we get three sets of postulates, with properties presenting very remarkable relations, which somewhat resemble those of the three geometries, Euclidean, Riemannian and Lobatchewskian. We shall enumerate them for convenience of comparison.

System (A).

(i). $A ⊜ A$.
(ii). If $A ⊜ B$, then $B ⊜ A$.
(iii). If $A ⊜ B$ and $B ⊜ C$, then $A ⊜ C$.

(iv). If $A ⧀ B$ and $B ⊜ C$, then $A ⧀ C$.
(v). If $A ⧁ B$, then $B ⧀ A$.
(vi). If $A ⧀ B$, then $B ⧁ A$.

System (B).

If A is a singular element of System (B), then the following propositions hold good.

(i). $A ⧁ A$.
(ii). If $A ⊜ B$, then $B ⧁ A$.
(iii). If $A ⊜ B$ and $B ⊜ C$, then $C ⧁ A$.

(iv). If $A \olessthan B$ and $B \circledequal C$, then $A \circledequal C$ or $A \olessthan C$.
(v). If $A \ogreaterthan B$, then $B \ogreaterthan A$ or $B \olessthan A$.
(vi). If $A \olessthan B$, then $B \ogreaterthan A$.

System (C).

If A is a singular element of System (C), then the following propositions hold good.

(i). $A \olessthan A$.
(ii). If $A \circledequal B$, then $B \olessthan A$.
(iii). If $A \circledequal B$ and $B \circledequal C$, then $C \olessthan A$.
(iv). If $A \ogreaterthan B$ and $B \circledequal C$, then $A \circledequal C$ or $A \ogreaterthan C$.
(v). If $A \olessthan B$, then $B \olessthan A$ or $B \ogreaterthan A$.
(vi). If $A \ogreaterthan B$, then $B \olessthan A$.

The following propositions are true for any element of the three systems; the former two as theorems and the latter two as postulates for Systems (A) and (B), and the former two as postulates and the latter two as theorems for System (C).

(i). If $A \olessthan B$ and $B \olessthan C$, then $A \olessthan C$.
(ii). If $A \circledequal B$ and $B \olessthan C$, then $A \olessthan C$.
(iii). If $A \ogreaterthan B$ and $B \ogreaterthan C$, then $A \ogreaterthan C$.
(iv). If $A \circledequal B$ and $B \ogreaterthan C$, then $A \ogreaterthan C$.

Ordered class of things (in a wider sense).

Definition. A class of things satisfying the following four postulates is called an ordered class of things (in a wider sense).

Postulate I. Any two elements of a class of things satisfy one, and only one, of the three relations $\circledequal$, $\ogreaterthan$, $\olessthan$.

Postulate III. If $A \olessthan B$, then $B \ogreaterthan A$.

Postulate IV. If $A \circledequal B$ and $B \ogreaterthan C$, then $A \ogreaterthan C$,

Postulate V. If $A \ogreaterthan B$ and $B \ogreaterthan C$, then $A \ogreaterthan C$.

We have already seen that elements of a class of things satisfying Postulates I, II′, III, IV, V may be divided into two classes, namely ordinary and singular elements. But among classes of things satisfying Postulates I, III, IV, V there may exist a class of things which contains, besides ordinary and singular elements, an element having no element equal to it, nor any to which it is equal. We shall give an example of such class of things.

Consider the Class ($\mathfrak{B}$) and add to it a number $N_1(200, 1)$. If we define the relations of the elements of this new class as in the class ($\mathfrak{B}$),

then this class satisfies Postulates I, III, IV, V, and contains the elements of the three kinds mentioned above For, the element $N(100, 1)$ has the element $A(99, 0)$ to which it is equal, but none which is equal to it; and the element $N_1(200, 1)$ has no element, to which it is equal, nor any which is equal to it. All the other elements are equal to themselves and so have elements to which they are equal and also elements which are equal to them. Thus the class contains elements of the three kinds. Further that this class satisfies Postulates I, III, IV, V may be proved in a manner similar to the proof of independence of Postulate II (section 2).

Definition. An element of a class of things, which has no element equal to it, nor any to which it is equal, is called an isolated element.

Thus, from this point of view, the elements of an ordered class of things may be classified into the three kinds:

(i) *ordinary element,*

(ii) *singular element,*

(iii) *isolated element.*

Theorem 31. In an ordered class of things, an ordinary element is equal to itself, and a singular element and an isolated element are greater than themselves.

The former part of this theorem may be proved as in the proof of Proposition I (section 4), and the latter part as in the proof of Theorem 23.

Theorem 32. Any ordered class of things containing a singular element always contains an ordinary element.

Proof. If A is a singular element of the class, it has an element B to which it is equal. Accordingly B must be equal to itself. For, if the relation $B \circledgtr B$ holds good, then from the relations $A \circleddash B$ and $B \circledgtr B$, the relation $A \circledgtr B$ would follow (Postulate IV), contrary to the relation $A \circleddash B$. Next if the relation $B \circledless B$ holds good, then we must have the relation $B \circledgtr B$ (Postulate III), which is again a contradiction. Therefore we have the relation $B \circleddash B$ and accordingly we have also the relations $A \circleddash B \circleddash B$, which shows that B is an ordinary element.

Cor. 1. There cannot exist an ordered class of things consisting of singular elements only.

Cor. 2. There cannot exist an ordered class of things consisting of singular and isolated elements only.

But there may exist a class of things consisting of isolated elements only. We shall give an example of it.

Consider a class of numbers $\{A(a, b)\}$, where a denotes any integer and b any number between zero and $\frac{1}{10}$ $\left(\text{excluding zero and } \frac{1}{10}\right)$, and define the relations of its elements as follows.

(i) If $A(a, b)$, $B(a', b')$ denote two elements of the class, then A is said to be equal to B when $a-a'=b+b'$, and (ii) A is said to be greater than B when $a-a'<b+b'$; and (iii) A is said to be less than B when $a-a'>b+b'$.

Now in this class of numbers any element A cannot be equal to any other element B and also to itself, since $a-a'$ is an integer and $b+b'$ is a positive proper fraction. Moreover there is no element equal to A by the same reason. Therefore A is an isolated element. Next this class of numbers satisfies Postulates I, III, V actually and Postulate IV vacuously, all of which can be verified at once.

As to the existence of an ordered class of things consisting of ordinary and singular elements only and also the existence of an ordered class containing all three kinds of elements, we have already given examples of them previously. Further, if we omit the element N from the class of things $\{N_1, N, A(a, b)\}$ mentioned before, we have an ordered class of things consisting of ordinary and isolated elements only; and if we take a class of things satisfying Postulates I, II, III, IV, V, we have an ordered class of things consisting of ordinary elements only. Thus we may classify the ordered classes of things as follows.

I. Classes containing only one kind of elements:

1. those which contain ordinary elements only,

2. those which contain isolated elements only.

II. Classes containing two kinds of elements:

1. those which contain ordinary and singular elements only,

2. those which contain ordinary and isolated elements only.

III. Classes containing three kinds of elements.

In this respect, the ordered classes of things present a very striking resemblance to the continuous set of points. In this Journal, Vol. 12, Nos. 1-2, I have shown that the elements of continuous sets of points may be classified into three kinds, namely, *non-principal point, simple principal point,* and *compound principal point;* and also have shown that continuous sets themselves may be classified by these points as follows.

I. Sets containing only one kind of points.

1. those which contain non-principal points only,

2. those which contain compound principal points only.

II. Sets containing two kinds of points:

1. *those which contain non-principal and compound principal points only,*
2. *those which contain non-principal and simple principal points only.*

III. Sets containing three kinds of points.

The sets containing simple principal points only or those containing simple and compound principal points only do not exist.

Thus we see that the ordinary, singular, and isolated elements of the ordered class of things just correspond to the non-principal, simple principal, and compound principal points of the continuous set of points respectively; and the classification of both sets by these elements are exactly the same.

SECTION 9. SECOND SINGULAR SET OF POSTULATES.

We have derived a singular set of postulates by replacing Postulate II of the first set of postulates in section 1 by Postulate II′. Now replace Postulate III "if $A \lessgtr B$, then $B \gtrless A$" of the same set of postulates by the analogous one "if $A \gtrless B$, then $B \lessgtr A$" (Postulate III′), then, from the new set, Postulate III does not follow, while, from the old one, Postulate III′ does. That Postulate III cannot be deduced from the new set of postulates may be seen from the following example.

Consider a system of particles moving in parallel straight lines and suppose that any straight line contains more than two particles moving in different directions, and define their equality and inequality as follows.

(i) If two particles A and B in the same line move in the same direction, then they are said to be equal to each other; and (ii) if A and B move in different lines and A's line is situated on the left of B's, then A is said to be greater than B; and (iii) in all other cases A is said to be less than B. We shall call this system of particles "the System $(\mathfrak{M})$".

According to the above definitions, it may be easily seen that this system of particles satisfies all the Postulates I, II, III′, IV, V, but not Postulate III; for in this system when A is less than B, B may be greater or less than A.

Moreover, there is a class of things satisfying these five postulates, in which all of the three cases $B(=)A$, $B(>)A$, $B(<)A$ may occur when the relation $A(<)B$ exists; for example, in the above system of particles, define the equality and inequality of its elements as follows.

(i) If two particles A and B in the same line move in the same direction and A is before B or A and B are at the same place, then A is said to be equal to B. (ii) (a) If two particles A and B in the same line move in the same direction and A is after B, or (b) if A and B move in the same line in opposite directions, or (c) if A and B move in different lines and A's line is situated on the left of B's, then A is said to be less than B. (iii) If two particles A and B move in different lines and A's line is situated on the right of B's, then A is said to be greater than B. We shall call this system of particles with the above definitions of equality and inequality "the System ($\mathfrak{N}$)".

According to these definitions, this system of particles clearly satisfies Postulates I, II, III′, IV, V, and when A is less than B, B is equal to or less than or greater than A, according as A and B move in the state of (a) or (b) or (c) respectively.

Thus the set of Postulates I, II, III′, IV, V embraces two main branches, in one of which Postulate III holds good while in the other it does not in general. The former branch is identical with the usual set of postulates, but the latter branch presents many properties different from the usual, whose striking feature is the indetermination of the relations of equality and inequality. We shall give some of them.

1. In a class of things satisfying Postulates I, II, III′, IV, V, but not Postulate III, no element can be greater than itself (Postulate III′), but it may be equal to or less than itself. We may construct a class of things containing two kinds of elements, one of which satisfies the relation $A \textcircled{=} A$, while the other satisfies the relation $B \textcircled{<} B$. For example, in a system of particles moving in parallel straight lines, in each of which more than two particles move in opposite directions, if we define the relations of its elements as follows, we get what we require.

(i) When both of the two particles A and B in the same line move upward or when they move in opposite directions, they are said to be equal to each other; and (ii) when both of the two particles A and B in the same line move downward, or when A and B move in different lines and A's line is situated on the left of B's, A is said to be less than B; and (iii) when A and B move in different lines and A's line is situated on the right of B's, A is said to be greater than B. We shall call this system of particles with the above definitions of equality and inequality "the System ($\mathfrak{D}$)".

According to these definitions, this system of particles satisfies Postulates I, II, III′, IV, V. And in this system, any element moving up-

ward is equal to itself; and any element moving downward is less than itself by the definitions of "equal to" and "less than".

2. In a class of things satisfying Postulates I, II, III′, IV, V, but not Postulates III, if the relation $A \circleddash B$ holds good, then the relation $B \circledgt A$ cannot (Postulate III′), but the other relations $B \circleddash A$ and $B \circledless A$ may both occur (see the System $(\mathfrak{N})$).

3. In the above class, if the relation $A \circledless B$ holds good, all of the three relations $B \circleddash A$, $B \circledgt A$, $B \circledless A$ may occur (see the System $(\mathfrak{N})$).

4. Similarly, in the above class, the relation of A to C is not uniquely determined in general when the relations of A to B and B to C are given (see the Systems $(\mathfrak{N})$ and $(\mathfrak{O})$). In general, the relation of A_1 to A_n is not uniquely determined when the relations of A_1 to A_2, A_2 to $A_3, \cdots\cdots, A_{n-1}$ to A_n are given.

Thus we see that classes of things satisfying Postulates I, II, III′, IV, V may be divided into two main classes presenting a very striking contrast in their properties; namely in one of them almost all the relations of equality and inequality are uniquely determined from its hypothesis (usual classes of things, the relations of their elements being defined in the usual manner), while, in the other, almost all the relations of equality and inequality are not uniquely determined from its hypothesis.

Here we shall give a very remarkable example of the latter kind.

A class of things satisfying five Postulates I, II, III′, IV, V, yet not satisfying all the fundamental theorems established by the usual set of Postulates I, II, III, IV, V.

Consider a system of particles moving in vertical coplanar parallel straight lines, and suppose that in every line there are particles moving in both directions, upward and downward, and also suppose that to any particle moving downward there is always a particle moving after it in the same line and in the same direction. Now in this system of particles, by the positions and directions of moving particles at instant t, we give the following definitions of the relations of these particles.

(i). A particle A is said to be in the relation $A \circleddash B$ with respect to B when they are in any one of the following states.

(a) The two particles A and B are moving in the same line and are approaching each other, A moving upward. (b) The two particles A and B are in the same line and both are moving downward, A being after B. (c) The two particles A and B are in the same line and both are moving upward, A being before B or coincident with B.

(ii). A particle A is said to be in the relation $A \circledless B$ with respect

to B when they are in any one of the following states.

(a′) The two particles A and B are moving in the same line and in opposite directions, (1) A and B meeting at a certain point, or (2) both going forward after they meet, or (3) A moving downward and approaching B. (b′) The two particles A and B are in the same line and both are moving downward, (1) A being before B, or (2) A being coincident with B. (c′) The two particles A and B are in the same line and both are moving upward, A being after B. (d′) The two particles A and B are in different lines and A's line is situated on the left of B's. (e′) The two particles A and B are in different lines and A's line is situated on the right of B's, and (1) A and B are moving in opposite directions, or (2) both are moving downward.

(iii). A particle A is said to be in the relation $A \ogreaterthan B$ with respect to B when they are in different lines and A's line is situated on the right of B's and both are moving upward.

According to these definitions, this system of particles clearly satisfies all five Postulates I, II, III′, IV, V. But in all cases other than those mentioned in the above postulates, the relation of the two elements is not uniquely determined from the given relations as will be seen below.

I. (i) If A is in the state (c), then it is in the relation $A \circledequal A$; and (ii) if A is in the state (b′), then it is in the relation $A \olessthan A$. Thus in this system each of the two relations $A \circledequal A$ and $A \olessthan A$ may occur.

II. (i) If A and B are in the state (c), then A and B are in the relation $A \circledequal B$ while B and A are in the relation $B \circledequal A$ or in the relation $B \olessthan A$; and (ii) if A and B are in the state (b), then A and B are in the relation $A \circledequal B$ while B and A are in the relation $B \olessthan A$. Thus in this system, from the relation $A \circledequal B$, each of the two relations $B \circledequal A$ and $B \olessthan A$ may follow.

III. (i) If A and B are in the state (d′), then A and B are in the relation $A \olessthan B$ while B and A are in the relation $B \ogreaterthan A$ or in the relation $B \olessthan A$; and (ii) if A and B are in the state (a′) (3), then A and B are in the relation $A \olessthan B$ while B and A are in the relation $B \circledequal A$. Thus in this system, from the relation $A \olessthan B$, each of the three relations $B \circledequal A$, $B \ogreaterthan A$, $B \olessthan A$ may follow.

IV. If A and B are in the state (c) and B and C are in the state (a), then A, B and C are in the relations $A \circledequal B$ and $B \circledequal C$, while A and C are in the relation $A \circledequal C$ or $A \olessthan C$. Thus, in this system, from the relations $A \circledequal B$ and $B \circledequal C$, each of the two relations $A \circledequal C$ and $A \olessthan C$ may follow.

V. (i) If A and B are in the state (a), and B and C are in the state (b′), then A, B and C are in the relations $A ⊜ B$ and $B ⧀ C$ while A and C are in the relation $A ⊜ C$; and (ii) if A and B are in the state (c) and B and C are in the state (d′), then A, B and C are in the relations $A ⊜ B$ and $B ⧀ C$ while A and C are in the relation $A ⧀ C$. Thus, in this system, from the relations $A ⊜ B$ and $B ⧀ C$, each of the two relations $A ⊜ C$ and $A ⧀ C$ may follow.

VI. (i) If A and B are in the state (e′) (1), A moving upward, and B and C are also in the state (e′) (1), then A, B and C are in the relations $A ⧀ B$ and $B ⧀ C$ while A and C are in the relation $A ⧁ C$; and (ii) if A and B are in the state (a′) (2), A moving upward, and B and C are also in the state (a′) (2), then A, B, and C are in the relations $A ⧀ B$ and $B ⧀ C$ while A and C are in the relation $A ⊜ C$ or in the relation $A ⧀ C$. Thus, in this system, from the relations $A ⧀ B$ and $B ⧀ C$, each of the three relations $A ⊜ C$, $A ⧁ C$ and $A ⧀ C$ may follow.

VII. (i) If A and B are in the state (iii) and B and C are in the state (c), then A, B and C are in the relations $A ⧁ B$ and $B ⊜ C$ while A and C are in the relation $A ⧁ C$; and (ii) if A and B are in the state (iii) and B and C are in the state (a), then A, B and C are in the relations $A ⧁ B$ and $B ⊜ C$ while A and C are in the relation $A ⧀ C$. Thus, in this system, from the relations $A ⧁ B$ and $B ⊜ C$, each of the two relations $A ⧁ C$ and $A ⧀ C$ may follow.

VIII. (i) If A and B are in the state (c′) and B and C are in the state (a), then A, B and C are in the relations $A ⧀ B$ and $B ⊜ C$ while A and C are in the relation $A ⊜ C$; and (ii) if A and B are in the state (b′) (1), and B and C are in the state (b), then A, B and C are in the relations $A ⧀ B$ and $B ⊜ C$ while A and C are in the relation $A ⊜ C$ or in the relation $A ⧀ C$. Thus, in this system, from the relations $A ⧀ B$ and $B ⊜ C$, each of the two relations $A ⊜ C$ and $A ⧀ C$ may follow.

IX. If A and B are in the state (d′) and B and C are in the state (iii), then A, B and C are in the relations $A ⧀ B$ and $B ⧁ C$ while A and C are in one of the relations $A ⊜ C$, $A ⧁ C$ and $A ⧀ C$. Thus, in this system, from the relations $A ⧀ B$ and $B ⧁ C$, each of the three relations $A ⊜ C$, $A ⧁ C$ and $A ⧀ C$ may follow.

X. If A and B are in the state (iii) and B and C are in the state (d′), then A, B and C are in the relations $A ⧁ B$ and $B ⧀ C$ while A and C are in one of the relations $A ⊜ B$, $A ⧁ C$ and $A ⧀ C$. Thus, in this system, from the relations $A ⧁ B$ and $B ⧀ C$, each of the three relations $A ⊜ C$, $A ⧁ C$ and $A ⧀ C$ may follow.

Thus in all of the above ten cases, which include all possible cases except those given in the postulates themselves, the relations of equality and inequality are not uniquely determined from their hypotheses, for in cases III, VI, IX, X, all three relations occur from the same given relations, and in all other cases two relations occur from the same given relations. We shall call this system of particles with the above definitions of equality and inequality "the System ($\mathfrak{P}$)".

When Postulates III′ of the set of Postulates I, II, III′, IV, V is replaced by Postulate III, in any class of things satisfying these postulates, the above indeterminations (except the cases IX, X) never occur. Whence we see what an important rôle one postulate plays in a set of postulates. The above example is a very interesting one to explain this fact.

Remark. In the above system of particles, when n elements are taken, and the relations of A_1 to A_2, A_2 to $A_3, \cdots\cdots$, A_{n-1} to A_n are given, the relation of A_1 to A_n is uniquely determined in none of the possible cases except in the two following ones only:

(i) $A_1 ⊜ A_2$, $A_2 ⧁ A_3$, $A_3 ⧁ A_4, \cdots\cdots$, $A_{n-1} ⧁ A_n$;

(ii) $A_1 ⧁ A_2$, $A_2 ⧁ A_3$, $A_3 ⧁ A_4, \cdots\cdots$, $A_{n-1} ⧁ A_n$.

In these two cases the relation of A_1 to A_n is uniquely determined as a necessary consequence of Postulates IV and V.

Now we shall determine whether Postulates I, II, III′, IV, V are independent of one another or not. This chiefly depends on whether the relation of an element to itself may be considered as meaningless or not. When the former consideration is taken and certain elements of a class of things in question are not repeated, Postulate III′ is independent of the others.

To prove it, consider the Class ($\mathfrak{B}$) and add to it the elements $N_n\,(100+2n, 1)$ and $B_n\,(100+\overline{2n-1}, 0)$ $(n=1, 2, 3, \cdots\cdot)$ and suppose that each element except N_n, B_n and $N_0\,(100, 1)$ is repeated any number of times. Now in this new class, define the relations ⊜, ⧁, ⧀ as in the Class ($\mathfrak{B}$), then this class of numbers satisfies Postulates I, II, IV, V, but not Postulate III′. The proof is as follows.

Firstly, by the definitions of the three relations the class clearly satisfies Postulate I.

Secondly, by the same definitions, we have the relations

$N_n ⊜ B_n$, $B_{n+1} ⊜ N_n$, $A_r ⊜$(its repeated element)(¹),

(¹) $A_r = A_r\,(100-r, 0)$ $(r=1, 2, 3, \cdots\cdot)$.

which shows that the class satisfies Postulate II.

Thirdly, when the relations $P(a, b) \circleddash Q(a', b')$ and $Q(a', b') \circledS R(a'', b'')$ hold good, we have

$$a - a' = b + b', \qquad a' - a'' < b' + b''$$

and accordingly $$a - a'' < b + b'' + 2b'.$$

Therefore if $b' = 0$, the relation $P \circledS R$ holds good and Postulate IV is satisfied. Thus to see that Postulate IV always holds good, we have only to examine the case $b' = 1$, namely where Q is one of the N's. In this case, from the relations $P \circleddash Q$ and $Q \circledS R$, we have

$$B_{n+1} \circleddash N_n, \quad N_n \circledS B_r \text{ or } N_n \circledS N_s \quad (r \gtreqless n+1, \ s \gtreqless n).$$

But since B and N occur only once and we do not consider the relation of an element to itself, $r = n + 1$[1] and $s = n$ are to be rejected, and so in the above case we have

$$B_{n+1} \circledS B_r \text{ and } B_{n+1} \circledS N_s \ (r > n+1, \ s > n),$$

which shows that in this case also Postulate IV is satisfied.

Fourthly, when the relations $P \circledS Q$, $Q \circledS R$ hold good, we have

$$a - a' < b + b', \qquad a' - a'' < b' + b'',$$

and accordingly $$a - a'' < b + b'' + 2b'.$$

Therefore to see that Postulate V always holds good, we have, as before, only to examine the case where Q is one of the N's. In this case, from the relations $P \circledS Q$, $Q \circledS R$, we have

$B_n \circledS N_n$ or $A_t \circledS N_n$, and $N_n \circledS B_r$ or $N_n \circledS N_s$ ($r, s \gtreqless n+1$, t any integer),

and from the definitions of the three relations we have

$$A_t \circledS B_r, \quad A_t \circledS N_s, \quad B_n \circledS B_r, \quad B_n \circledS N_s,$$

which shows that Postulate V is also satisfied in this class.

Lastly, when two elements N_n and B_n are taken, we have the relation $B_n \circledS N_n$, while, on the other hand, the relation $N_n \circleddash B_n$ holds good which shows that Postulate III′ is not satisfied in this class.

For the convenience of reference we shall call this class "the Class $(\mathfrak{Q})$." Thus from the Class $(\mathfrak{Q})$ we may conclude that Postulate III′ is independent of the others. Next that each of Postulates I, II, IV, V is independent of the others may be proved by using the Class $(\mathfrak{A})$,

---

(1) In the theory of equality and inequality, when three elements A, B and C are given, we have to determine the relation of A to C from the two given relations of A to B and B to C. Therefore, if the relation of an element to itself is not admitted, the three elements considered must be necessarily distinct from one another.

($\mathfrak{B}'$), ($\mathfrak{D}$), and ($\mathfrak{E}'$) respectively. Thus when the former consideration is taken, all five postulates are independent of one another. But, on the other hand, when the latter consideration is taken, or when all the elements of a class are repeated, we may prove Postulate III′ from the other postulates as will be seen below.

Lemma. Any element of a class satisfying Postulates I, II, IV, V cannot be greater than itself or its repeated one.

For, suppose that, if possible, the relation $A \ogreaterthan A$ hold good, then since, by Postulate II, there is an element B such that $B \oequal A$, from the relations $B \oequal A$ and $A \ogreaterthan A$ we should have the relation $B \ogreaterthan A$ uniquely (Postulate IV), contrary to the relation $B \oequal A$. Thus the relation $A \ogreaterthan A$ cannot hold good.

From this lemma, we may prove that the relation $B \olessthan A$ always holds good whenever the relation $A \ogreaterthan B$ does. For, suppose that, if possible, the relation $B \oequal A$ holds good in this case, then from the relations $B \oequal A$ and $A \ogreaterthan B$ we should have the relations $B \ogreaterthan B$ uniquely (Postulate IV), contrary to the above lemma. Next suppose that, if possible, the relation $B \ogreaterthan A$ hold good, then from the relations $B \ogreaterthan A$ and $A \ogreaterthan B$, again we should have the relation $B \ogreaterthan B$ (Postulate V), contrary to the lemma. Thus, by Postulate I, we must have the relation $B \olessthan A$.

Since, in the Class ($\mathfrak{P}$) satisfying Postulates I, II, III′, IV, V, no definite conclusion is obtained from any one of the ten hypotheses there given, we may conclude from the above discussion that, as far as three or less elements of a class are concerned, the number of definite fundamental propositions deduced from the set of Postulates I, II, IV, V is at most only one(1); and even when n elements are taken, the number of definite propositions deducible is at most only three. But, when the latter consideration mentioned above is taken, by the addition of only one suitable postulate (Postulate III) almost all these indeterminations at once cease and we get at one stroke a great many definite propositions concerning them. *This is a very striking example to show that the addition of only one postulate has great influence on a set of postulates to establish a branch of science.*

In passing, we shall give here another remarkable example of the same kind. In the preceding, we have seen that, when the relation of

(1) The proposition is "if $A \ogreaterthan B$, then $B \olessthan A$."

an element to itself is considered as meaningless and certain elements of a class of things are not repeated, no definite proposition is deduced from the set of Postulates I, II, IV, V, and, if otherwise, only one definite proposition is deduced from the same set as far as three or less elements are concerned ; and also we have remarked that in the latter case the addition of only one postulate makes almost all the propositions definite. But, on the other hand, in the former case, the addition of the same postulate (Postulate III) make only two propositions definite and leave all the others indefinite. The two definite propositions deducible are the following :

(i) if $A ⧀ B$ and $B ⧀ C$, then $A ⧀ C$,

(ii) if $A ⊜ B$ and $B ⧀ C$, then $A ⧀ C$.

First that these propositions are true may be proved at once. Next that all the others are left as indefinite may be seen from the Class $(\mathfrak{Q})$ and the Class $(\mathfrak{R})$. It was already proved that the Class $(\mathfrak{Q})$ satisfies Postulates I, II, IV, V ; and moreover, that it also satisfies Postulate III may be seen at once from the definitions of the relations $⧀$, $⧁$. Now in this Class $(\mathfrak{Q})$, we have the following indeterminations.

I. In this class, the relation of the element N to itself is meaningless while the element A_r is in the relation of $A_r ⊜ A_r'$ to another repeated element A_r'.

II. In this class, N_n and B_n are in the relation $N_n ⊜ B_n$ while B_n and N_n are in the relation $B_n ⧁ N_n$; and further A_r and its repeated element A_r' are in the relation $A_r ⊜ A_r'$ while A_r' and A_r are in the relation $A_r' ⊜ A_r$. Thus, in this class, from the relation $A ⊜ B$, each of the two relations $B ⊜ A$ and $B ⧁ A$ may follow.

III. In this class, B_n and N_n are in the relation $B_n ⧁ N_n$ while N_n and B_n are in the relation $N_n ⊜ B_n$; and further A_r and A_{r-1} are in the relation $A_r ⧁ A_{r-1}$ while A_{r-1} and A_r are in the relation $A_{r-1} ⧀ A_r$. Thus, in this class, from the relation $A ⧁ B$, each of the two relations $B ⊜ A$ and $B ⧀ A$ may follow.

IV. In this class, B_1 and N_0, N_0 and A_1 are in the relations $B_1 ⊜ N_0$ and $N_0 ⊜ A_1$ while B_1 and A_1 are in the relation $B_1 ⧀ A_1$; and further N_n and B_n, B_n and N_{n-1} are in the relations $N_n ⊜ B_n$ and $B_n ⊜ N_{n-1}$ while N_n and N_{n-1} are in the relation $N_n ⊜ N_{n-1}$. Thus, in this class, from the relations $A ⊜ B$ and $B ⊜ C$, each of the two relations $A ⧀ C$ and $A ⊜ C$ may follow.

V. In this class, A_1 and N_0, N_0 and A_1' (respeated element of A_1) are in the relations $A_1 ⧁ N_0$ and $N_0 ⊜ A_1'$, while A_1 and A_1' are in the

relation $A_1 ⊜ A_1'$; and further A_r and A_{r-1}, A_{r-1} and A'_{r-1} are in the relations $A_r ⧁ A_{r-1}$ and $A_{r-1} \dot{⊜} A'_{r-1}$ while A_r and A'_{r-1} are in the relation $A_r ⧁ A'_{r-1}$. Thus, in this class, from the relations $A ⧁ B$ and $B ⊜ C$, each of the two relations $A ⊜ C$ and $A ⧁ C$ may follow.

VI. In this class, the three elements A_p, A_q, A_r are in the relations $A_p ⧁ A_q$, $A_q ⧀ A_r$ when $p>q$ and $r>q$, while A_p, A_r are in the relation $A_p ⊜ A_r$, or $A_p ⧁ A_r$, or $A_p ⧀ A_r$ according as $p=r$, or $p>r$, or $p<r$. Thus, in this class, from the relations $A ⧁ B$ and $B ⧀ C$, each of the three relations $A ⊜ C$, $A ⧁ C$ and $A ⧀ C$ may follow.

VII. Similarly from the relations $A ⧀ B$ and $B ⧁ C$, each of the three relations $A ⊜ C$, $A ⧁ C$ and $A ⧀ C$ may follow.

Lastly, to see that from the relations $A ⧀ B$ and $B ⊜ C$, each of the two relations $A ⊜ C$ and $A ⧀ C$ may follow, consider a class of numbers $3, 1, 0, -3, -6, \cdots$, no element being repeated, and define the relations of its elements as follows.

If $A(a)$, $\mathrm{B}(b)$ denote two elements of the class, then A is said to be in the relation $A \dot{⊜} B$ with respect to B when one of the two equalities $a+3=b$, $a\pm 2=b$ holds good; and A is said to be in the relation $A ⧁ B$ with respect to B when $a>b$ $(a-2\neq b)$; and A is said to be in the relation $A ⧀ B$ with respect to B when $a<b$ $(a+3\neq b,$ $a+2\neq b)$. (Since no element is repeated and the relation of an element to itself is meaningless in this class, the case $a=b$ never occurs.)

According to these definitions, the class satisfies Postulate I, and moreover, since we have the relations

$$A_2(1) ⊜ A_1(3), \qquad A_3(0) ⊜ A_1(3), \qquad A_1(3) ⊜ A_2(1),$$
$$A_{n+1}(-3n+6) ⊜ A_n(-3n+9) \qquad (n=3, 4, 5, \cdots\cdots),$$

from the above definitions, the class also satisfies Postulate II.

Next when A and B are in the relation $A(a) ⧀ B(b)$, we have the inequality $a<b$ and accordingly the inequality $b>a$. In this case the equality $b-2=a$ never occurs since there are only two elements $A_1(3)$, $A_2(1)$ satisfying it and they are in the relations $A_1 ⊜ A_2$ and $A_2 ⊜ A_1$. Therefore when the relation $A ⧀ B$ holds good, we have always the relation $B ⧁ A$, which shows that Postulate III is also satisfied in this class.

Further, when A, B and C are in the relations $A ⊜ B$ and $B ⧁ C$, (i) if A represents A_1, then B must be A_2, and C one of $A_3, A_4, \cdots\cdots$, and so in this case A and C are in the relation $A ⧁ C$. (ii) If A represents A_2, then B must be A_1, and C one of A_3, $A_4, \cdots\cdots$, and so in this case

also A and C are in the relation $A \textcircled{>} C$. (iii) If A represents A_3, then B must be A_1, and C one of A_4, $A_5, \cdots\cdots$ (1), and so in this case also A and C are in the relation $A \textcircled{>} C$. (iv) If A represents A_n $(n=4, 5, \cdots\cdots)$, then B must be A_{n-1}, and C one of A_{n+1}, $A_{n+2}, \cdots\cdots$, and so in this case also A and C are in the relation $A \textcircled{>} C$. Thus Postulate IV is also satisfied in this class.

Lastly, when A, B and C are in the relations $A(a) \textcircled{>} B(b)$ and $B(b) \textcircled{>} C(c)$, we have $a > c$ from the definition of the relation $\textcircled{>}$, and moreover, in this case, $a-2=c$ never occurs, since, if it did, a, b, c would be in the series $c+2$, $c+1$, c, which is impossible in our class. So we have always the relation $A \textcircled{>} C$ from the relations $A \textcircled{>} B$ and $B \textcircled{>} C$.

Thus the above class of numbers satisfies all the five postulates in question. We call this class of numbers "the Class ($\mathfrak{N}$)."

VIII. In the above class, if the three elements $A_1(3)$, $A_2(1)$, and $A_3(0)$ are taken, then they are in the relations $A_3 \textcircled{<} A_2$, and $A_2 \textcircled{=} A_1$, while A_3 and A_1 are in the relation $A_3 \textcircled{=} A_1$. Next if the three elements $A_5(-6)$, $A_3(0)$ and $A_1(3)$ are taken, then they are in the relations $A_5 \textcircled{<} A_3$ and $A_3 \textcircled{=} A_1$, while A_5 and A_1 are in the relation $A_5 \textcircled{<} A_1$. Thus, in this class, from the relations $A \textcircled{<} B$ and $B \textcircled{=} C$, each of the two relations $A \textcircled{=} C$ and $A \textcircled{<} C$ may follow.

Therefore, in a class of things satisfying Postulates I, II, III, IV, V, the above eight indeterminations may occur in the fundamental propositions concerning three relations, and the definite propositions deducible from these postulates are only two. But if the relation of an element to itself is admitted to have a meaning or if Postulate II is replaced by the postulate "$A \textcircled{=} A$," then all these indeterminations (except VI, VII) at once cease, and six definite propositions are obtained in virtue of only one postulate. *Thus here we have an example which is suitable to show that the replacement of only one postulate in a set of postulates has a great effect in obtaining many theorems.*

PART III.

Categorical Sets of Independent Postulates.

Though the set of postulates given in section 1 is sufficient to deduce all the propositions concerning equality and inequality, yet two

(1) See the foot note on p. 258.

classes of things satisfying these postulates are not necessarily isomorphic with respect to the three relations $(\equiv)$, $(>)$, $(<)$; in other words, there are various classes of things which satisfy all these postulates, but whose elements cannot be brought into a one-to-one correspondence in such a way that, whenever two elements a and b in one class correspond to a' and b' in the other class, they are always in the same relation. In other words, the above set of postulates is not categorical. To obtain a categorical set of postulates we have to add others to the above. Here we shall discuss such sets of postulates.

I. Categorical Set of Postulates concerning the Three Undefined Relations.

(First Type).

Before proceeding to give a categorical set of postulates, we shall give some definitions required in the following discussion.

Definition I. In any class of things, if three distinct elements A, B, and X satisfy the relation $A(<)X$ and $X(<)B$, then X is said to lie *between* A and B.

Definition II. In any class of things, if two distinct elements A and X satisfy the relation $A(<)X$ and no element exist between A and X, then X is called *an immediate successor of* A. Similarly, if the relation $X(<)A$ holds good and no element exists between X and A, then X is called *an immediate predecessor of* A. If A and B satisfy the relation $A(<)B$, then A is called *a predecessor* of B, *and* B *a successor of* A.

Definition III. In any class of things, if there exists an element X having no predecessor, then the element X is called *the first element of the class.* Similarly, the element X having no successor is called *the last element of the class.*

A Categorical Set of Postulates.

I. *Any two elements A and B of a class of things satisfy at least one of the three relations $A(\equiv)B$, $A(>)B$, $A(<)B$.*

II. *Any element A of the class has n, and only n, elements B_r $(r=1, 2, \cdots\cdots, n)$, satisfying the relation $B_r(\equiv)A$.*

III. *If $A(<)B$, then $B(>)A$.*

IV. *If $A(\equiv)B$ and $B(>)C$, then $A(>)C$.*

V. *If $A(>)B$ and $B(>)C$, then $A(>)C$.*

VI. The class has the first element.

VII. Any element of the class has its immediate successor.

VIII. Any two distinct elements A and B of the class, satisfying the relation $A \lessdot B$, are related by a finite number of the relation $\lessdot$, so that

$$A \lessdot M_1 \lessdot M_2 \lessdot \cdots\cdots \lessdot M_n \lessdot B,$$

where M_1 is an immediate successor of A, and M_2 that o M_1, and so on.

Categoricalness of the Set of Postulates.

Theorem. Every element of the class, unless it is the first, has an immediate predecessor.

Proof. Denote the first element by A, and any other element by P, then by Postulate VIII, A and P are related by the following relations

$$A \lessdot M_1 \lessdot M_2 \lessdot \cdots\cdots\cdots\cdots \lessdot M_l \lessdot P,$$

so that P has an immediate predecessor M_l.

Theorem. Any two distinct elements A and B, satisfying the relation $A \gtrdot B$, are related by a finite number of the relation $\gtrdot$, so that

$$A \gtrdot M_1 \gtrdot M_2 \gtrdot \cdots\cdots\cdots\cdots \gtrdot M_n \gtrdot B,$$

where any two successive elements M_s and M_{s+1} have no element satisfying the relation $M_s \gtrdot X \gtrdot M_{s+1}$.

Proof. From the relation $A \gtrdot B$, we have the relation $B \lessdot A$ by Postulates I, II, III, IV, V, and so we have the relations

$$B \lessdot N_1 \lessdot N_2 \lessdot \cdots\cdots\cdots\cdots \lessdot N_l \lessdot A$$

by Postulate VIII. But by Postulate III and the above relations we have

$$A \gtrdot N_l,\ N_l \gtrdot N_{l-1}, \cdots\cdots\cdots,\ N_1 \gtrdot B,$$

and between any two successive elements N_s and N_{s+1} of the above relations there is no element X satisfying the relations $N_{s+1} \gtrdot X \gtrdot N_s$, since, if there were, N_{s+1} would not be an immediate successor of N_s. Therefore we have the required relations

$$A \gtrdot N_l \gtrdot N_{l-1} \gtrdot \cdots\cdots\cdots\cdots \gtrdot N_1 \gtrdot B.$$

Theorem. If A is an immediate successor (or an immediate predecessor) of B, then any element equal to A is an immediate successor (or an immediate predecessor) of any element equal to B.

Proof. Take any elements A_p and B_q equal to A and B respectively, then from the relations $B_q \circleddash B$ and $B \circledless A_p$, we have the relation $B_q \circledless A$; and from the relations $B_q \circledless A$ and $A \circleddash A_p$ we have the relation $B_q \circledless A_p$. Moreover, there is no element between B_q and A_p, since, if there were, there would also be an element between B and A, contrary to the hypothesis. Therefore A_p is an immediate successor of B_q.

Theorem. Any classes of things satisfying the above eight postulates can be brought into a one-to-one correspondence, such that, in that correspondence, corresponding elements of the classes are always in the same relation.

Proof. Take any two classes of things satisfying the above postulates and denote them by $\{A\}$ and $\{B\}$. Taking any elements A_p and B_q from them, denote the n elements equal to them by A_{p_1}, $A_{p_2}, \cdots\cdots, A_{p_n}$ and B_{q_1}, $B_{q_2}, \cdots\cdots, B_{q_n}$ respectively. Now taking any one of $(n+1)$ first elements, say A_o, from $\{A\}$, assign it to any one of $(n+1)$ first elements of $\{B\}$, say B_o, and A_{o_1} to B_{o_1} and A_{o_2} to B_{o_2} and so on. Next, taking any one of $(n+1)$ immediate successors of the first element from $\{A\}$, assign it to any one of $(n+1)$ immediate successors of the first element of $\{B\}$, and any remaining one of the above successors in the class $\{A\}$ to any remaining one of the successors in the class $\{B\}$ and so on. Proceeding in this way, we get the required one-to-one correspondence, since, by Postulate VIII, no element of either class will be inaccessible to this process. Thus the set of our postulates is categorical.

Independence of the Postulates.

I. Postulate I is independent of the other postulates.

Consider a class of natural numbers with zero, and add to it a class consisting of all couples $\{(a, b)\}$, where a represents 1 and b any natural number; and suppose that every element of these two classes is repeated m times. Now in the class of natural numbers, define the three relations $\circleddash$, $\circledgtr$, $\circledless$ as "equal to," "greater than" and "less than" in the meaning commonly used; and in the class $\{(a, b)\}$, define them according as the sum of a, b of an element $A(a, b)$ is equal to, or

greater than, or less than[1], that of another element; and moreover, define that, for every element of $\{(a, b)\}$, the relations $0 \circledless (a, b)$ and $(a, b) \circledgtr 0$ always hold good. Then in the class consisting of the above two classes, the relation of any element of the class of natural numbers (except 0) to any element of the class $\{(a, b)\}$ is left undefined, but this class satisfies all the postulates other than Postulate I. Their verifications may be easily done.

II. Postulate II is independent of the other postulates.

Consider a class of natural numbers, all greater than 3 being repeated m times, and the three numbers 1, 2, 3 not being repeated. In this class of numbers define the three relations $\circleddash$, $\circledgtr$, $\circledless$ as above, then this class of numbers satisfies all the postulates other than Postulate II. Their verifications may be easily done.

III. Postulate III is independent of the other postulates.

Consider a class of natural numbers, no one of its elements being repeated, and define the relations of its elements as follows.

If $A(a)$ and $B(b)$ denote two elements of the class, then A and B are said to be in the relation $A \circleddash B$ when $a-b=1$; and they are said to be in the relation $A \circledless B$ when $a-b<1$; and they are said to be in the relation $A \circledgtr B$ when $a-b>1$. Then this class of numbers satisfies all the postulates other than Postulate III.

Proof. That this class satisfies Postulates I, IV, V may be seen at once, and moreover, since every element has one, and only one, element which is equal to it, namely its succeeding one, Postulate II is also satisfied. But when we take two neighbouring elements $A(a)$, $B(b)$, we have the relation $A \circledless B$ while B and A are in the relation $B \circleddash A$. Thus Postulate III does not hold good in this class. Further any element $A(a)$ has $B(b=a+1)$ as its immediate successor. For, if there were an element $X(x)$ between A and B, we should have the relation $A \circledless X \circledless B$ and accordingly the two inequalities

$$a-x<1, \qquad x-b<1;$$

but, since three elements A, B and X are distinct and as any one of them occurs only once in the class, from $x-b<1$ we must have $x<b$. Again, since $b=a+1$, from $x<b$ we must have $x<a$ for the same reason as above, which contradicts the hypothesis $a-x<1$. Therefore each element of the class, when they are arranged in the order of their

---

(1) They are taken in the meaning commonly used.

values, has its succeeding element as its immediate successor. Hence it follows at once that Postulates VI, VII, VIII are all satisfied in this class of numbers, the element $A(1)$ being the first element.

IV. Postulate IV is independent of the other postulates.

Consider a class consisting of the following numbers, none of them being repeated,

$$A_0(1),\ B(2),\ A_1(5),\ A_2(9),\cdots\cdots,\ A_n(1+4n),\cdots\cdots$$

and define the relations of its elements as follows.

If $A(a)$ and $B(b)$ denote two elements of the class, then A and B are said to be in the relation $A \circleddash B$ when $a+b=3$, or $a+4=b$, or $a-b=0$; and they are said to be in the relation $A ⧀ B$ when $a<b$ $(a+b\neq 3,\ a+4\neq b)$; and they are said to be in the relation $A ⧁ B$ when $a>b$ $(a+b\neq 3)$.

According to these definitions, the class satisfies all the postulates other than Postulate IV.

Proof. That the class satisfies Postulates I, III may be seen at once, and moreover by the definition of the relation $\circleddash$, we have

$$(1)\qquad \begin{cases} A_0 \circleddash B \\ B \circleddash A_0 \\ A_{n-1} \circleddash A_n (n=1, 2, 3, 4,\cdots\cdots) \end{cases}$$

Thus every element of the class has one, and only one, element which is equal to it, and so Postulate II is also satisfied. Further, when we take any three elements $A(a)$, $B(b)$, $C(c)$ which are in the relations $A ⧁ B$ and $B ⧁ C$, we have

$$a>b \qquad (a+b\neq 3),$$
$$b>c \qquad (b+c\neq 3),$$

and accordingly we have

$$a>c \qquad (a+c\neq 3),$$

which shows that A and C are in the relation $A ⧁ C$. Thus Postulate V is also satisfied. But when we take the three elements A_0, A_1, B we have the relations $A_0 \circleddash A_1$ and $A_1 ⧁ B$ while A_0 and B are in the relation $A_0 \circleddash B$. Therefore Postulate IV does not hold good in this class.

Further, from the relation (1), we may easily deduce that B has A_1, A_2 as its immediate successors, and A_m $(m=0, 1, 2, 3,\cdots\cdots)$ has A_{m+2}, A_{m+3} as its immediate successors. Whence we see that Postulates VI, VII, VIII are also satisfied in this class of numbers, each of A_1 and B being the first element of the class.

V. Postulate V is independent of the other postulates.

Consider the class consisting of the following numbers, each number being repeated m times,

(1) $A_1(3, 0),\ A_2(3, 2),\ \cdots\cdots,\ A_n(3, 2n-2),\cdots\cdots$

(2) $B_1(1, 1),\ B_2(1, 3)\ \cdots\cdots,\ B_n(1, 2n-1),\cdots\cdots,$

and define the relations of its elements as follows.

If $P(a, b)$, and $Q(a', b')$ denote two elements of the class, (i) when $a-a'=0$, P and Q are said to be in the relation $P \circleddash Q$, or $P \circledgtr Q$, or $P \circledless Q$ according as $b=b'$, or $b>b'$, or $b<b'$ respectively. (ii) When $a-a'\neq 0$, if P belongs to $\{A\}$, then Q belongs to $\{B\}$ and vice versâ; in this case, (a) if $a+b$ is even and the inequality $a+b<a'+b'$ holds good, or if $a+b$ is odd and the inequalities $a+b<a'+b'>a+2b$ hold good, then P and Q are said to be in the relation $P \circledless Q$; (b) next if $a+b$ is odd and the inequalities $a+b<a'+b'<a+2b$ hold good, or if the inequality $a+b>a'+b'$ holds good, then P and Q are said to be in the relatiou $P \circledgtr Q$(1). According to these definitions, this class of numbers satisfies all the postulates other than Postulate V.

Proof. That the class satisfies Postulates I, II, III, IV may be easily seen, and when we take the three elements $A(3, 2)$, $B(1, 5)$ and $C(3, 2)$, we have the relations $A \circledgtr B$ and $B \circledgtr C$ while A and C are in the relation $A \circleddash C$. Thus the class does not satisfy Postulate V.

Further, in this class, $B_1(1, 1)$ is the first element, since, if not, there would exist an element $X(a, b)$, such that $a+b<1+1$, but this is impossible. Thus Postulate VI is also satisfied. Next to show that Postulate VII is also satisfied take any element A_n from $\{A\}$, then by the definitions of the relations $\circledgtr$, $\circledless$, we have the following relations.

$$(1)\quad \begin{cases} A_{n-1} \circledless A_n \circledless A_{n+1}, \\ B_{n-1} \circledless B_n \circledless B_{n+1}; \end{cases}$$

$$(2)\quad \begin{cases} B_n \circledless A_n \circledless B_{2n}, \\ A_n \circledgtr B_{n+1},\ A_n \circledgtr B_{n+2}, \cdots\cdots, A_n \circledgtr B_{2n-1}\ (n>1); \end{cases}$$

$$(3)\quad \begin{cases} A_{n+1} \circledgtr B_{2n}\ (n>1), \\ B_{2n} \circledgtr A_{n+1}\ (n>1), \text{ and } B_{2n} \circledless A_{n+1}\ (n=1). \end{cases}$$

Moreover we know that the relations $P \circledless Q$ and $Q \circledless P$ cannot exist at the same time by Postulate III. Therefore from the above relations we may conclude that there is no element lying between A_n and A_{n+1}

(1) In the second case the equality $a+b=a'+b'$ never occurs.

$(n>1)$, and also between A_n and B_{2n} $(n \gtreqless 1)$. Thus, by definition, when $n>1$, A_{n+1} and B_{2n} are immediate successors of A_n; and, when $n=1$, B_2 is an immediate successor of A_1. Next take any element B_n from $\{B\}$, then we have the following relations

$$(1) \quad \begin{cases} B_{n+1} \circledless B_n \circledless B_{n+1}, \\ \qquad B_n \circledless A_n, \ A_{n-1} \circledgtr B_n, \ B_n \circledgtr A_{n-1}; \end{cases}$$

$$(2) \quad \begin{cases} A_n \circledgtr B_{n+1} \ (n>1), \text{ and } A_n \circledless B_{n+1} \ (n=1), \\ B_{n+1} \circledgtr A_n. \end{cases}$$

Therefore, in this case, when $n>1$, B_n has B_{n+1} and A_n as its immediate successors; and when $n=1$, B_1 has A_1 as its immediate successor. Thus Postulate VII is also satisfied in this class.

From the above discussion it follows that in each of the series

$$(1) \qquad A_1, \ A_2, \ A_3, \cdots\cdots$$

$$(2) \qquad B_1, \ B_2, \ B_3, \cdots\cdots$$

the former of any two neighbouring elements is an immediate predecessor of the latter, and the latter an immediate successor of the former. Moreover B_n has A_n as its immediate successor while A_n has B_{2n} as its immediate successor. Whence we may easily infer that the class also satisfies Postulate VIII.

VI. Postulate VI is independent of the other postulates.

Consider a class of all integers, every number being repeated m times, and define the three relations $\circleddash$, $\circledgtr$, $\circledless$ as "equal to" "greater than" and "less than" in the meaning commonly used. Then it may be seen at once that this class of numbers satisfies all the postulates other than Postulate VI.

VII. Postulate VII is independent of the other postulates.

Consider a class of n natural numbers $1, 2, 3, \cdots\cdots, n$, every element being repeated m times, and define the three relations as in VI, then it may be seen at once that this class of numbers satisfies all the postulates other than Postulate VII. But since the element n has no immediate successor, the class does not satisfy Postulate VII.

VIII. Postulate VIII is independent of the other postulates.

Consider a class of the following finite and transfinite numbers,

$$1, 2, 3, \cdots\cdots, \ \omega, \ \omega+1, \ \omega+2, \cdots\cdots,$$

every element being repeated m times, and define the three relations as above, then it may be seen at once that this class of numbers satisfies all the postulates other than Postulate VIII. But when we take $A(1)$

and $B(\omega)$, we cannot reach B from A by finite steps, so Postulate VIII is not satisfied by this class.

Consistency of the Postulates.

The consistency of the eight postulates may be seen at once by taking all natural numbers, each element being repeated m times, and by defining the three relations $\ominus$, $\circledgtr$, $\circledless$ as above, for, in this class of numbers all eight postulates are satisfied.

Other Categorical Sets of Postulates belonging to the Same Type as above.

If we replace Postulate VI by any one of the following three postulates :—

Postulate VI′. The class has the last element

Postulate VI″. The class has the first and last elements

Postulate VI‴. The class has neither the first nor the last element,

and if we make a corresponding slight modification in Postulate VII, then we have three categorical sets of postulates belonging to the same type as above.

II. Categorical Sets of Postulates concerning the Three Undefined Relations.

(Second Type).

In the above, we have obtained the four categorical sets of postulates corresponding to the class of integers. Now we proceed to discuss categorical sets of postulates corresponding to the class of rational numbers.

A Set of Postulates.

I. *Any two elements A and B of a class of things satisfy at least one of the three relations $\ominus$, $\circledgtr$, $\circledless$.*

II. *Any element A of the class has n, and only n, elements B_r $(r=1, 2, 3, \cdots\cdots, n)$ satisfying the relation $B_r \ominus A$.*

III. *If $A \circledless B$, then $B \circledgtr A$.*

IV. *If $A \ominus B$ and $B \circledgtr C$, then $A \circledgtr C$.*

V. *If $A \circledgtr B$ and $B \circledgtr C$, then $A \circledgtr C$.*

VI. *If $A \circledless B$, then there is at least one element X satisfying the relation $A \circledless X \circledless B$, A, B, X being distinct elements of the class.*

VII. The number of elements of the class is denumerable.
VIII. The class has neither the first nor the last element.

Categoricalness of the Postulates.

Since every element of a class of things satisfying the above eight postulates has n, and only n, elements equal to it, we can divide the elements of the class into $(n+1)$ sets by putting one, and only one, of $(n+1)$ equal elements to every one of these sets. Then the elements of these sets can be put in a one-to-one correspondence when we make the elements correspond which are in the relation $\ominus$ to each other. Now, by Postulates I, II, III, IV, V, the corresponding elements of these sets are in the same relation, namely the elements of each set are ordinally similar[1] to those of any other set; and, moreover, the elements of every set satisfy Postulates I, II, III, IV, V, VI, VII, VIII.

To prove the categoricalness of the postulates take any two classes of things $\{H\}$, $\{H'\}$, satisfying the above eight postulates; and divide the elements of each of them into $(n+1)$ sets as mentioned above, and denote them by $\{H_1\}$, $\{H_2\}, \cdots\cdots$, $\{H_{n+1}\}$; $\{H_1'\}$, $\{H_2'\}, \cdots\cdots$, $\{H_{n+1}'\}$. Then by using the properties of the sets $\{H_1\}$ and $\{H_1'\}$, we can prove that the elements of the two sets $\{H_1\}$, $\{H_1'\}$ can be brought into a one-to-one correspondence, such that the corresponding elements are in the same relation, by exactly the same reasoning as Huntington used in proving ordinal similarity of two series of the type η[2]. Hence it follows at once that all elements of the two classes $\{H\}$ and $\{H'\}$ can also be brought into the isomorphic state with respect to the three relations $\ominus$, $\textcircled{>}$, $\textcircled{<}$.

Independence of the Postulates.

I. Postulate I is independent of the other postulates.

Consider two classes of numbers, each consisting of all rationals—call them red and blue—and suppose that every element of the class is repeated m times; and define the relations $\ominus$, $\textcircled{>}$, $\textcircled{<}$ of the elements belonging to the same class as "equal to," "greater than" and "less than" in the meaning commonly used. But the relation of any red element to any blue element is left undefined. Then the class consisting of these two classes does not satisfy Postulate I while it satisfies all the other postulates. Their verification may be shown at once.

(1) The words "ordinally similar" is used here in the meaning given by Huntington in his "The Continuum"

(2) Huntington, The Continuum, p. 35.

II. Postulate II is independent of the other postulates.

Consider a class of all rationals and suppose that every element of the class (except the element 1) is repeated m times, and define the three relations as above, then it may be seen at once that this class of numbers satisfies all the postulates other than Postulate II.

III. Postulate III is independent of the other postulates.

Consider a class of all couples $\{(a, b)\}$, where a denotes any integer and b any positive rational, and suppose that every element is repeated m times, and define the relations of its elements as follows.

If $A(a, b)$ and $B(a', b')$ denote two elements of the class, then A is said to be in the relation $\ominus$ with respect to B when, and only when, $a=a'$, $b=b'$; and the remaining cases are divided into two and A is said to be in the relation $\circledgt$ with respect to B when $a-a' \gtreqless b+b'$; and A is said to be in the relation $\circledless$ with respect to B when $a-a'<b+b'$ (except in the case $a=a'$, $b=b'$). Then this class satisfies all the postulates other than Postulate III.

Proof. That this class satisfies Postulates I, II, IV, V may be easily seen and when we take four elements a, a', b, b', such that

$$|a-a'|<b+b' \text{ (}^{1}\text{)},$$

we have the relations

$$A(a, b) \circledless B(a', b') \quad \text{and} \quad B(a', b') \circledless A(a, b)$$

by the definition of the relation $\circledless$. So Postulate III is not satisfied by this class of numbers.

Next to show that Postulate VI is also satisfied by this class, we have to show that, when two elements $A(a, b)$ and $B(a', b')$ which are in the relation $A \circledless B$ are given, there always exists at least one element $X(m, n)$, such that

$$A(a, b) \circledless X(m, n) \circledless B(a', b'),$$

that is, we have to show that there are two numbers m, n, such that

(i) $\qquad a-m<b+n,$

(ii) $\qquad m-a'<n+b' \quad$ (m : integer, n : positive rational).

But this is always possible, since we can take n as great as we please and accordingly we can take n, so that it is greater than $a-m-b$ and $m-a'-b'$ (m may denote any integer).

Further, since the numbers of integers and rationals are denumerable

(1) It is clear that such numbers exist in the above class.

and a class composed of a denumerable infinity of denumerable classes is itself denumerable, Postulate VII is also satisfied by this class of numbers. Moreover, since we can always find X and X_1, such that

$$A(a, b) \lessdot X(m, n),$$

$$X_1(m_1, n_1) \lessdot A(a, b),$$

when any element A is given, the class has neither the first nor the last element, so Postulate VIII is also satisfied by this class.

IV. Postulate IV is independent of the other postulates.

Consider a class of all rationals and suppose that no element is repeated; and define the relations of its elements as follows.

If $A(a)$ and $B(b)$ denote two elements of the class, then A is said to be in the relation $\circleddash$ with respect to B when $a+1=b$; and A is said to be in the relation $\lessdot$ with respect to B when $a<b$ and $a+1 \neq b$; and A is said to be in the relation $\gtrdot$ with respect to B when $a>b$[1].

According to these definitions, this class satisfies all the postulates other than Postulate IV.

Proof. That this class satisfies Postulates I, III, V, VII, VIII may be seen at once, and when any element $B(b)$ is given there is one, and only one, element $A(a)$ satisfying the equality $a+1=b$. Therefore the class also satisfies Postulate II. But when we take three elements $A(a)$, $B(a+1)$ and $C\left(a+\frac{1}{2}\right)$, we have the relations $A \circleddash B$, $B \gtrdot C$ and $A \lessdot C$, so that the class does not satisfy Postulate IV.

Further, when two elements $A(a)$ and $B(b)$ are given, so that they are in the relation $A(a) \lessdot B(b)$ and accordingly a, b satisfy the inequalities $a<b$, $a+1 \neq b$, we can always find a number $X(x)$ satisfying the inequalities

$$a<x<b, \quad a+1 \neq x, \quad x+1 \neq b.$$

Therefore Postulate VI is also satisfied by this class.

V. Postulate V is independent of the other postulates.

Consider the Class ($\mathfrak{C}$), and in that class take rational numbers a', c' instead of real numbers a, c, and take a positive rational number b' instead of a positive real number b, and suppose that every element is repeated m times. Then that this class satisfies Postulates I, II, III, IV,

---

[1] The case $a=b$ can never occur since no element is repeated; but if it is desired to know the relation of an element to itself we may define that in this case A is in the relation $A \circleddash A$.

but not Postulate V may be proved in a manner similar to the proof of independence of Postulate V in section 2.

Further, to prove that, when two elements A and B satisfying the relation $A(a', b', c') \lessdot B(a_1', b_1', c_1')$ are given, there is always an element $X(\alpha, \beta, \gamma)$ satisfying the relation

$$A \lessdot X \lessdot B,$$

we have to find α, β, γ, satisfying the two inequalities

$$\text{(i)} \qquad a' - \alpha + b' + \beta < c' - \gamma,$$

$$\text{(ii)} \qquad \alpha - a_1' + \beta + b_1' < \gamma - c_1'.$$

From (i) and (ii), we have

$$\text{(iii)} \qquad c' - a' + \alpha - \beta - b' > \gamma > c_1' + \alpha - a_1' + \beta + b_1'$$

and accordingly (iv) $c' - a' + \alpha - b' - \beta > c_1' + \alpha - a_1' + \beta + b_1'$.
From (iv), it follows that the value of β is to be determined, so that it satisfies the inequalities

$$\text{(v)} \qquad (c' - c_1') - (a' - a_1' + b' + b_1') > 2\beta.$$

But, since, by hypothesis, the relation $A \lessdot B$ holds good, we have

$$a' - a_1' + b' + b_1' < c' - c_1'$$

or

$$(c' - c_1') - (a' - a_1' + b' + b_1') > 0.$$

Therefore we have to find a positive rational value of β less than a given positive number

$$\frac{(c' - c_1') - (a' - a_1' + b' + b_1')}{2},$$

which is always possible. With this value of β thus determined, find the rational value of γ satisfying the relation (iii), giving to α any rational value. Then the three numbers α, β, γ thus determined have the required property. So Postulate VI is also satisfied by this class.

Next that the number of elements (a', b', c') is denumerable may be seen from the fact that a class composed of a denumerable infinity of denumerable classes is also denumerable; and that the class has neither the first nor the last element may be seen from the fact that we can always find the numbers $B(a_2', b_2', c_2')$ and $C(a_3', b_3', c_3')$ satisfying the inequalities

$$a_1' - a_2' + b_1' + b_2' < c_1' - c_2',$$

$$a_3' - a_1' + b_3' + b_1' < c_3' - c_1'$$

when any number $A\,(a_1', b_1', c_1')$ is given.

VI. Postulate VI is independent of the other postulates.

Take all integers, every number being repeated m times, and define the three relations ⊜, ⧁, ⧀ as "equal to," "greater than" and "less than" in the meaning commonly used, then it may be seen at once that this class satisfies all the postulates other than Postulate VI.

VII. Postulate VII is independent of the other postulates.

Take all real numbers, every number being repeated m times, and define the three relations ⊜, ⧁, ⧀ as above, then it may be seen at once that this class satisfies all the postulates other than Postulate VII.

VIII. Postulate VIII is independent of the other postulates.

Take all rational numbers not less than 1, every number being repeated m times, and define the three relations ⊜, ⧁, ⧀ as above, then it may be seen at once that this class satisfies all the postulates other than Postulate VIII.

Consistency of the Postulates.

The consistency of the postulates may be seen at once by taking all rational numbers, every number being repeated m times, and by defining the three relations ⊜, ⧁, ⧀ as "equal to", "greater than" and "less than" in the meaning commonly used, for, in this class of numbers all eight postulates are satisfied.

Other Categorical Sets of Postulates belonging to the Same Type as above.

If we replace Postulate VIII by any one of Postulates VI, VI′, VI″ of the first type, then we have three other categorical sets of postulates belonging to the same type as above.

III. Categorical Sets of Postulates concerning Three Undefined Relations (Third Type).

Lastly we shall give categorical sets of postulates corresponding to the class of real numbers.

A Set of Postulates.

I. Any two elements of a class of things satisfy at least one of the three relations ⊜, ⧁, ⧀.

II. *Any element of the class has n, and only n, elements equal to it.*

III. *If $A \circledless B$, then $B \circledgtr A$.*

IV. *If $A \circledequal B$ and $B \circledgtr C$, then $A \circledgtr C$.*

V. *If $A \circledgtr B$ and $B \circledgtr C$, then $A \circledgtr C$.*

VI. *(Dedekind Postulate). If $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are any two non-empty parts of a class $\mathfrak{K}$, such that every element of $\mathfrak{K}$ belongs either to $\mathfrak{K}_1$ or to $\mathfrak{K}_2$ and every element K_1 of the part $\mathfrak{K}_1$ is in the relation $K_1 \circledless K_2$ to every element K_2 of the part $\mathfrak{K}_2$, then there is at least one element X in $\mathfrak{K}$, such that*

(1) any element P which is in the relation $P \circledless X$ belongs to $\mathfrak{K}_1$ and (2) any element Q which is in the relation $X \circledless Q$ belongs to $\mathfrak{K}_2$.

VII. *(Postulate of Linearity). The class $\mathfrak{K}$ contains a denumerable subclass $\mathfrak{R}$ in such a way that, corresponding to any two elements A, B of the given class $\mathfrak{K}$, there is an element X of $\mathfrak{R}$ satisfying the relation $A \circledless X \circledless B$, when A, B are in the relation $A \circledless B$.*

VIII. *The class has neither the first nor the last element.*

Categoricalness of the Postulates.

As in the classes of things belonging to the second type, any class of things satisfying the eight postulates of the third type can be divided into $(n+1)$ sets, the elements of every set being ordinally similar to those of any other sets. Moreover the elements of any one of these sets clearly satisfies the sets of postulates with which Huntington defines a series of the type θ(1). So these elements can be brought into a one-to-one correspondence to the elements of the class of real numbers, such that the corresponding elements are in the same relation.

Now take any two classes of things $\mathfrak{K}$ and $\mathfrak{K}'$ satisfying the eight postulates mentioned above; and divide the elements of each of them into $(n+1)$ sets, then by the above property, the elements of any one set of the one class may be brought into a one-to-one correspondence maintaining ordinal similarity to elements of any one set of the other class. Hence it follows at once that the elements of the two classes $\mathfrak{K}$ and $\mathfrak{K}'$ can also be brought into the isomorphic state with respect to the three relations $\circledequal$, $\circledgtr$, $\circledless$.

(1) Huntington, The Continuum, p. 49.

Independence of the Postulates.

I. Postulate I is independent of the other postulates.

Consider two classes of all real numbers—call them red and blue—and suppose that every element of the classes is repeated m times, and define the relations of their elements as follows.

If $A(a)$ and $B(b)$ denote any elements of the two classes, then they are said to be in the relation $A \circledgt B$ or in the relation $A \circledless B$ according as $a>b$ or $a<b$; and if they belong to the same class (red or blue), then they are said to be in the relation $A \circleddash B$ when $a=b$; and if they belong to different classes and moreover if $a=b$, then their relation is left undefined.

According to these definitions it may be seen at once that the class composed of these classes satisfies Postulates II, III, IV, V, VIII, but not Postulate I. Further, to see that the class also satisfies Postulate VI, suppose that it is divided into two non-empty classes $\mathfrak{K}_1$ and $\mathfrak{K}_2$ having the property stated in Postulate VI, then, by the property of real numbers and the above definitions of the relations $\circledgt$, $\circledless$, there is at least one element $X(x)$, such that all elements having the values less[1] than x belong to $\mathfrak{K}_1$ and those having the values greater than x to $\mathfrak{K}_2$, and those having the value equal to x to either $\mathfrak{K}_1$ or $\mathfrak{K}_2$. This element $X(x)$ has the required property, and so Dedekind Postulate is satisfied in this class.

Next if we take a class of all rational numbers as the subclass $\mathfrak{R}$, then it may be easily seen that, when $A \circledless B$, there is an element X of $\mathfrak{R}$ lying between A and B. Thus Postulate VII is also satisfied in this class.

II. Postulate II is independent of the other postulates.

Consider a class of all real numbers, every number of the class (except 1) being repeated m times, and define the three relations $\circleddash$, $\circledgt$, $\circledless$ as "equal to", "greater than" "less than" in the meaning commonly used, then it may be seen at once that this class satisfies all the postulates other than Postulate II.

III. Postulate III is independent of the other postulates.

Take an unlimited straight line and through all points of this line draw parallel coplanar straight lines perpendicular to the line; and suppose that in every line (except that which is drawn through the point

---

(1) The words "less than", "equal to", "greater than" are taken here in the meaning commonly used.

corresponding to the real number zero, which we may call the zero line) one and only one particle is moving. Further suppose that on the zero line there is a circle perpendicular to the plane of parallel lines and touching the zero line at zero point, and in this circle five points P', Q', R', S_1', S_2' are moving in the same derection with the same velocity. Next consider a plane parallel to the above, and suppose that on this plane the lines and particles of the above have their images.

Now consider a system of particles moving in the above two planes and define their relations as follows.

If one of the two particles A and B is the image of the other, then they are said to be in the relations $A \circledequal B$ and $B \circledequal A$; and if one of them is not the image of the other and A's line (straight line or circle) is situated on the left of B's, then A is said to be in the relation $\circledgtr$ with respect to B; and if A's line is situated on the right of B's, then A is said to be in the relation $\circledless$ with respect to B. Further, as to the relations of the particles in the circle to one another, we lay down the following definitions

$$
\begin{array}{llllll}
S_1 \circledless P, & P \circledgtr S_1; & S_2 \circledgtr P, & P \circledless S_2; & S_1 \circledless S_2, & S_2 \circledgtr S_1; \\
S_1 \circledless Q, & Q \circledgtr S_1; & S_2 \circledgtr Q, & Q \circledless S_2; & & \\
S_1 \circledless R, & R \circledgtr S_1; & S_2 \circledgtr R, & R \circledless S_2; & & \\
P \circledless Q, & Q \circledless R, & R \circledless P, & & & \\
P \circledless R, & Q \circledless P, & R \circledless Q\,(^1). & & &
\end{array}
$$

Here P denotes either P' or its image P''. Similarly for Q, R and S. Thus, of the above five points P, Q, R, S_1, S_2, S_1 is the first element while S_2 is the last element.

According to these definitions, this system satisfies all the postulates other than Postulate III.

Proof. That this system of particles satisfies Postulates I, II, IV, V, VIII may be easily seen, and that it does not satisfy Postulate III may be seen by taking two particles P and Q in the circle which are in the relations $P \circledless Q$ and $Q \circledless P$. Further, to see that this system of particles also satisfies Postulate VI, suppose that the system is divided into two non-empty parts $\mathfrak{K}_1$ and $\mathfrak{K}_2$ having the property stated in Postulate VI; then by the property of real numbers and the above definitions of the relations $\circledgtr$, $\circledless$, there is one line corresponding to a real number x,

(1) This class of particles P, Q, R, S and their images has an interesting property. At the end of this part, we shall give another example of this kind.

such that all the particles in the lines corresponding to real numbers greater than x belong to $\mathfrak{K}_2$ and those corresponding to real numbers less than x belong to $\mathfrak{K}_1$. If this line is straight, the particle X in this line and its image X' must both belong to the same subclass $\mathfrak{K}_1$ or $\mathfrak{K}_2$ and each of them has the property required in Dedekind Postulate. If this line is the circle, then all the particles P', Q', R' in this line and their images P'', Q'', R'' must belong to one, and only one, of the subclasses $\mathfrak{K}_1$ and $\mathfrak{K}_2$ by the rule of division. If they belong to $\mathfrak{K}_1$, then also S_1' and S_1'' must belong to $\mathfrak{K}_1$; and S_2' and S_2'' may belong to $\mathfrak{K}_1$ or $\mathfrak{K}_2$, and each of them has the property required in Dedekind Postulate. If P', Q', R' and their images belong to $\mathfrak{K}_2$, then S_2' and S_2'' must belong to $\mathfrak{K}_2$; and S_1' and S_1'' may belong to $\mathfrak{K}_1$ or $\mathfrak{K}_2$, and each of them has the required property. Thus, by this system, Dedekind Postulate is also satisfied.

Next, to see that Postulate VII is also satisfied by this system, take all the particles moving in the lines corresponding to rational numbers as the subclass $\mathfrak{R}$, then it is clear that this subclass $\mathfrak{R}$ is denumerable. Now when we take any two particles A, B which are in the relation $A \mathbin{ⓛ} B$, if both of them belong to the circle, since we have the relations

$$
\begin{array}{lll}
S_1 ⓛ Q ⓛ P, & P ⓛ Q ⓛ S_2, & S_1 ⓛ P ⓛ S_2, \\
S_1 ⓛ R ⓛ Q, & Q ⓛ R ⓛ S_2, & \\
S_1 ⓛ P ⓛ R, & R ⓛ P ⓛ S_2, & \\
P ⓛ R ⓛ Q, & Q ⓛ P ⓛ R, & R ⓛ Q ⓛ P, \\
P ⓛ Q ⓛ R, & Q ⓛ R ⓛ P, & R ⓛ P ⓛ Q,
\end{array}
$$

and moreover, since P, Q, R, S belong to the subclass $\mathfrak{R}$, we see that Postulate VII is satisfied in this case.

If A and B belong to different lines, we see at once that there is an element R_1 in the subclass $\mathfrak{R}$, such that it satisfies the relation $A ⓛ R_1 ⓛ B$. Thus Postulate VII is satisfied in this case also.

IV. Postulate IV is independent of the other postulates.

Consider a class composed of the four classes of all real numbers—call them red, blue, white and black—and define the relations of its elements as follows.

If $A(a)$ and $B(b)$ denote two numbers of the class, then A is said to be in the relation ⓖ with respect to B when $a > b$, and in the relation $A ⓛ B$ when $a < b$. When $a = b$, the relation of the two elements is laid down by the following:

white ⊜ red,	black ⊜ red,
red ⊜ white,	red ⊜ black,
blue ⊜ white,	blue ⊜ black,
white ⊜ blue,	black ⊜ blue,
red ⧁ blue,	red ⧁ red,
blue ⧁ red,	blue ⧁ blue,
white ⧁ black,	white ⧁ white,
black ⧁ white,	black ⧁ black.

Then this class of numbers satisfies all the postulates other than Postulate IV.

Proof. That this class satisfies Postulates I, II, III, V, VII, VIII may be seen at once, and when we take three elements, white, blue, red, having the same value, we have the relations

white ⊜ blue, blue ⧁ red, white ⊜ red,

so that Postulate IV is not satisfied by this class.

Further, to prove that the class also satisfies Postulate VI, suppose that the class is divided into two non-empty classes $\mathfrak{K}_1$, $\mathfrak{K}_2$ having the property stated in Postulate VI, then, by the property of real numbers and the definitions of the relations ⧁, ⧀, there is a real number x, such that all elements having a value less than x belong to $\mathfrak{K}_1$ and those having a value greater than x belong to $\mathfrak{K}_2$, and in this case all four elements having the same value x belong to one of the two classes $\mathfrak{K}_1$, $\mathfrak{K}_2$. Any one of these four elements has the property required in Dedekind Postulate, so Postulate VI is satisfied by this class.

V. Postulate V is independent of the other postulates.

Consider a class composed of four classes of all real numbers—call two of them red, and the other two of them blue—and define the relations of its elements as follows.

If $A(a)$ and $B(b)$ are two numbers of the class, then A is said to be in the relation ⧁ with respect to B when $a>b$, and in the relation A⧀B when $a<b$. When $a=b$, the relation of the two elements is laid down by the following:

red ⊜ red,	red ⧁ blue,
blue ⊜ blue,	blue ⧁ red.

Then that this class satisfies Postulates I, II, III, IV, VII, VIII may be seen at once, and moreover that it also satisfies Postulate VI

may be seen as in the case IV. But when we take two red and one blue elements having the same value, we have the following relations

first red $\circledgt$ blue, blue $\circledgt$ second red, first red $\circleddash$ second red.

Thus Postulate V is not satisfied by this class.

VI. Postulate VI is independent of the other postulates.

Consider a class of all rational numbers, every element being repeated m times; and define the relations $\circleddash$, $\circledgt$, $\circledlt$, as "equal to", "greater than" and "less than" in the meaning commonly used; and take the class of all rationals not repeated as subclass $\mathfrak{R}$, then it may be seen at once that this class satisfies all the postulates other than Postulate VI.

VII. Postulate VII is independent of the other postulates.

Consider a class of all couples $\{(a, b)\}$, where a and b are real numbers, and suppose that every element is repeated m times; and define the relations of its elements as follows.

If $A(a, b)$ and $B(a', b')$ denote two elements of the class, then A is said to be in the relation $\circleddash$ with respect to B when $a=a'$ and $b=b'$; and A is said to be in the relation $\circledgt$ with respect to B when $a>a'$, or when $a=a'$ and $b>b'$; and A is said to be in the relation $\circledlt$ with respect to B when $a<a'$, or when $a=a'$ and $b<b'$.

According to these definitions, the class satisfies Postulates I, II, III, IV, V, VI, VIII. But to obtain a subclass $\mathfrak{R}$ having the property stated in Postulate VII we must take all couples $\{(a, b)'\}$, such that the aggregate $\{a\}$ of $\{(a, b)'\}$ contains all real numbers. For, if the subclass $\mathfrak{R}$ contains no element, whose value of a is equal to a real number a_1, then $\mathfrak{R}$ would contain no element lying between the two elements (a_1, b') and (a_1, b'') of the class. Therefore the subclass $\mathfrak{R}$ cannot be denumerable; so Postulate VII is not satisfied by this class.

VIII. Postulate VIII is independent of the other postulates.

Consider a class of all positive real numbers including zero, every element being repeated m times, and define the equality and inequality of its elements as usual, then it may be seen at once that this class satisfies all the postulates other than Postulate VIII.

Consistency of the Postulates.

The consistency of the postulates may be seen at once by taking a class of all real numbers, every element being repeated m times, and by defining the three relations $\circleddash$, $\circledgt$, $\circledlt$ as "equal to", "greater than", and "less than" in the meaning commonly used.

Further if we replace Postulate VIII by any one of Postulates VI, VI′, VI″ of the first type, we have the other three categorical sets of postulates belonging to the same type as above.

At the end of this part, we shall add an interesting example of a class of things, the number of whose elements is finite, while it satisfies the postulate of density (Postulate VI of the second type).

When any class of things satisfying Postulates I, II, III, IV, V also satisfies Postulate VI of the second type, there will be at least one and therefore an infinity of elements between any two elements of the class, which are not in the relation $\circledequal$; accordingly any element of this class has neither an immediate successor nor an immediate predecessor, so that this class is said to be dense. We shall call this Postulate VI "postulate of density". But when we take Postulate III′ "if $A \circledgt B$, then $B \circledless A$" instead of Postulate III "if $A \circledless B$, then $B \circledgt A$", and consider a class of things satisfying Postulates I, II, III′, IV, V, VI, it may be seen at once that, in this case also, there is at least one element between any two elements, which are not in the relation $\circledequal$, so that no element of this class has either an immediate predecessor or an immediate successor. But as to the number of elements of the class, there occurs a striking difference between the preceding class and this one, for, the number of elements of the former is always infinite while that of the latter may be finite or infinite. An example of the latter class, the number of whose elements is finite, has been already given in the proof of the independence of Postulate III of the third type. We shall give another example, the number of whose elements may be finite or infinite.

Consider a class consisting of couples $S_1(0, 0)$, $S_2(200, 0)$ and $A_b(100, b)$ $(0<b<100)$, all elements being repeated once; and define the relations of equality and inequality as follows.

If $A(a, b)$ and $B(a', b')$ denote two elements of the class, then A and B are said to be in the relation $A \circledequal B$ when $a=a'$ and $b=b'$; and they are said to be in the relation $A \circledgt B$ when $a-a' \gtreqless b+b'$ (except the case $a=a'$, $b=b'$); and they are said to be in the relation $A \circledless B$ when $a-a'<b+b'$ (except the case $a=a'$, $b=b'$).

Then it may be seen at once that this class of numbers satisfies Postulates I, II, III′, IV, V. But this class does not satisfy Postulate III, for, take any two numbers b' and b'' lying between 0 and 100, and

construct the two numbers $A_{b'}(100, b')$ and $A_{b''}(100, b'')$ of the class, then we have the relations

$$A_{b'} \lessdot A_{b''}, \qquad A_{b''} \lessdot A_{b'} \tag{1}$$

by the definition of the relation $\lessdot$.

Moreover, S_1, S_2 and A_b are in the relations

$$\begin{aligned} &S_1 \lessdot A_b, && A_b \gtrdot S_1; \\ &S_2 \gtrdot A_b, && A_b \lessdot S_2; \\ &S_1 \lessdot S_2, && S_2 \gtrdot S_1. \end{aligned} \tag{2}$$

From (1) and (2), it follows at once that, when the class contains at least three elements of A_b's, there is always at least one lying between any two elements which are in the relation $\lessdot$; that is, the class satisfies the postulate of density.

Further there is no element X satisfying either of the relations

$$X \lessdot S_1, \qquad S_2 \lessdot X.$$

Therefore S_1 is the first element and S_2 the last element of the class.

Now in this class, (1) if we give to b the values of natural numbers lying between 0 and 100, we have a class of couples, the number of whose elements is finite; and (2) if we give to b the values of rational numbers lying between 0 and 100, we have a class, the number of whose elements is denumerably infinite; and (3) if we give to b the values of real numbers lying between 0 and 100, we have a class, the number of whose elements is non-denumerably infinite.

On the Fourier Constants,

by

KINNOSUKE OGURA, Osaka.

I.

Prof. A. Hurwitz proved the following theorem(1): If in the interval $(0 \leqq x \leqq 2\pi)$ the function $f(x)$ be finite and integrable and if all its Fourier constants be zero, then $f(x)$ is zero at every point of the interval at which it is continuous.

Now I will prove the following theorem:

Let $f(x)$ be any function which is absolutely integrable (or, more generally, integrable in the sense of Lebesgue) in the interval $(0 \leqq x \leqq 2\pi)$ and is such that the corresponding Fourier series

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\cos nx\int_0^{2\pi} f(t)\cos nt\,dt + \sin nx\int_0^{2\pi} f(t)\sin nt\,dt\right]$$

converges uniformly to the function $f(x)$ at every point of the interval $(a \leqq x \leqq b$, where $0 \leqq a < b < 2\pi$ or $0 < a < b \leqq 2\pi)$ at which the function is continuous. Then there exists either (i) none or (ii) an infinite number of the inequalities

$$\int_a^b f(t)\cos mt\,dt \neq 0,$$

$$\int_a^b f(t)\sin nt\,dt \neq 0,$$

where m and n are integers.

And in the case (i) the function $f(x)$ is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous(2).

---

(1) A. Hurwitz, Über die Fourierschen Konstanten integrierbarer Funktionen, Math, Ann., 57 (1903), p. 425. For a generalization of this theorem, see Steklov, Su la théorie de fermeture des systèmes de fonctions orthogonales, Mém. de l'Acad. St. Pétersbourg, (8) 30 (1911), p. 27.

(2) Prof. C. N. Moore proved that if in the interval $(0 \leqq a \leqq x \leqq b)$ $f(x)$ is finite save for a finite number of points, and is integrable and if

$$\int_a^b f(t)\cos nt\,dt = 0 \ \ (n=0, 1, 2, \cdots), \quad \int_a^b f(t)\sin nt\,dt = 0 \ \ (n=1, 2, \cdots),$$

Suppose that there is a finite number of positive integers $m_1, m_2, \cdots\cdots, m_p$; $n_1, n_2, \cdots\cdots, n_q$ such that

$$\int_a^b f(t) \cos m_\mu t\, dt \equiv A_{m_\mu} \neq 0 \qquad \text{for } \mu=1, 2, \cdots\cdots, p\text{(}^{1}\text{)};$$

$$\int_a^b f(t) \cos mt\, dt = 0 \qquad \text{for all non-negative integers } m \text{ except } m_1, m_2, \cdots\cdots, m_p;$$

$$\int_a^b f(t) \sin n_\nu t\, dt \equiv B_{n_\nu} \neq 0 \qquad \text{for } \nu=1, 2, \cdots\cdots, q;$$

$$\int_a^b f(t) \sin nt\, dt = 0 \qquad \text{for all positive integers } n \text{ except } n_1, n_2, \cdots\cdots, n_q,$$

where $0<a<b<2\pi$.

If we consider the function $\psi(x)$ defined by

$$\begin{aligned} \psi(x) &= 0 & 0 &\leqq x < a, \\ &= f(x) & a &\leqq x \leqq b, \\ &= 0 & b &< x \leqq 2\pi, \end{aligned}$$

then we see that the Fourier series corresponding to $\psi(x)$ converges uniformly to this function at every point of the interval $(0 \leqq x \leqq 2\pi)$ at which it is continuous([2]). But since

$$\int_0^{2\pi} \psi(t) \cos m_\mu t\, dt = A_{m_\mu} \qquad \text{for } \mu=1, 2, \cdots\cdots, p;$$

$$\int_0^{2\pi} \psi(t) \cos mt\, dt = 0 \qquad \text{for all non-negative integers } m \text{ except } m_1, m_2, \cdots\cdots, m_p;$$

$$\int_0^{2\pi} \psi(t) \sin n_\nu t\, dt = B_{n_\nu} \qquad \text{for } \nu=1, 2, \cdots\cdots, q;$$

$$\int_0^{2\pi} \psi(t) \sin nt\, dt = 0 \qquad \text{for all positive integers } n \text{ except } n_1, n_2, \cdots\cdots, n_q,$$

in the interval $(0 \leqq x \leqq 2\pi)$ the Fourier series corresponding to $\psi(x)$ converges uniformly to the function $S(x)$, where

$$S(x) = \frac{1}{\pi} \sum_{\mu=1}^{p} A_{m_\mu} \cos m_\mu x + \frac{1}{\pi} \sum_{\nu=1}^{q} B_{n_\nu} \sin n_\nu x;$$

---

then $f(t)$ is zero at every point of the interval $(a \leqq x \leqq b)$ at which it is continuous. See C. N. Moore, On a certain constants analogous to Fourier's constants, Bull. Amer. Math. Soc., 14 (1908), p. 371.

(1) If $m_\mu = 0$, then A_{m_μ} should be replaced by $\dfrac{A_0}{2}$.

(2) Vallée Poussin, Cours d'analyse infinitésimale, t. 2, (2. éd. 1912), p. 144.

so that

$$S(x)-\psi(x)=\frac{1}{\pi}\sum_{\mu=1}^{p} A_{m_\mu}\cos m_\mu x+\frac{1}{\pi}\sum_{\nu=1}^{q} B_{n_\nu}\sin n_\nu x \qquad 0\leqq x<a,$$

$$=\frac{1}{\pi}\sum_{\mu=1}^{p} A_{m_\mu}\cos m_\mu x+\frac{1}{\pi}\sum_{\nu=1}^{q} B_{n_\nu}\sin n_\nu x-f(x) \qquad a\leqq x\leqq b,$$

$$=\frac{1}{\pi}\sum_{\mu=1}^{p} A_{m_\mu}\cos m_\mu x+\frac{1}{\pi}\sum_{\nu=1}^{q} B_{n_\nu}\sin n_\nu x \qquad b<x\leqq 2\pi.$$

Since $\psi(x)$ is continuous in the intervals $(0\leqq x<a)$ and $(b<x\leqq 2\pi)$, we must have

$$\sum_{\mu=1}^{p} A_{m_\mu}\cos m_\mu x+\sum_{\nu=1}^{q} B_{n_\nu}\sin n_\nu x=0$$

at every point of the intervals $(0\leqq x<a)$ and $(b<x\leqq 2\pi)$; consequently

$$A_{m_\mu}=0 \qquad (\mu=1,\ 2,\cdots\cdots,\ p),$$

$$B_{n_\nu}=0 \qquad (\nu=1,\ 2,\ \cdots,\ q).$$

And therefore

$$S(x)-\psi(x)=-f(x) \qquad \text{for} \quad a\leqq x\leqq b.$$

But since

$$S(x)=\psi(x)$$

at every point of the interval $(a\leqq x\leqq b)$ at which $\psi(x)$ is continuous. Hence we must have

$$f(x)=0$$

at every point of the interval $(a\leqq x\leqq b)$ at which $f(x)$ is continuous.

It is evident that these results hold good for the cases $0=a,\ b<2\pi$; and $0<a,\ b=2\pi$.

The method of proof may be applied to the case of developments in terms of any other normal functions, such as the Legendre polynomials(1), the Sturm-Liouville functions, etc.(2), whenever we know that the series corresponding to any discontinuous function which satisfies a certain condition converges uniformly to that function at every point at which it is continuous.

(1) Hobson, On the representation of a function by a series of Legendre's functions, Proc. London Math. Soc.. (2) 7 (1908), p. 24.

(2) For example, see Kneser, Die Theorie der Integralgleichungen und die Darstellung willkürlicher Funktionen in der mathematischen Physik, Math. Ann., 63 (1907), p. 477; Kneser, Die Integralgleichungen (1911); Juretzka, Die Entwickelung unstetiger Funktionen nach den Eigenfunktionen des schwingenden Stabes auf Grund der Theorie der Integralgleichungen, Diss. Breslau (1909).

II.

Let $f(x)$ be the function defined by the series

$$(1) \qquad a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

which converges uniformly at every point of the interval $(0\leqq x\leqq 2\pi)$.

Take any two fixed numbers a and b within the interval $(0,\ 2\pi)$, and form the $2n+1$ rowed determinant

$$\Delta_n=\begin{vmatrix}
\int_a^b dx & \int_a^b \cos x\,dx & \int_a^b \sin x\,dx & \cdots\cdots & \int_a^b \cos nx\,dx & \int_a^b \sin nx\,dx \\
\int_a^b \cos x\,dx & \int_a^b \cos^2 x\,dx & \int_a^b \cos x\sin x\,dx & \cdots\cdots & \int_a^b \cos x\cos nx\,dx & \int_a^b \cos x\sin nx\,dx \\
\int_a^b \sin x\,dx & \int_a^b \sin x\cos x\,dx & \int_a^b \sin^2 x\,dx & \cdots\cdots & \int_a^b \sin x\cos nx\,dx & \int_a^b \sin x\sin nx\,dx \\
\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\
\int_a^b \cos nx\,dx & \int_a^b \cos nx\cos x\,dx & \int_a^b \cos nx\sin x\,dx & \cdots\cdots & \int_a^b \cos^2 nx\,dx & \int_a^b \cos nx\sin nx\,dx \\
\int_a^b \sin nx\,dx & \int_a^b \sin nx\cos x\,dx & \int_a^b \sin nx\sin x\,dx & \cdots\cdots & \int_a^b \sin nx\cos nx\,dx & \int_a^b \sin^2 nx\,dx
\end{vmatrix},$$

and let $\Delta_n^{(r)}$ be the determinant obtained from Δ_n by replacing the elements

of the $(r+1)$*-th column by the elements*

$$\int_a^b f(x)\,dx,\quad \int_a^b f(x)\cos x\,dx,\quad \int_a^b f(x)\sin x\,dx,\cdots\cdots$$
$$\cdots\cdots,\quad \int_a^b f(x)\cos nx\,dx,\quad \int_a^b f(x)\sin nx\,dx.$$

Then

$$\lim_{n=\infty}\frac{\Delta_n^{(o)}}{\Delta_n}=a_0,$$
$$\lim_{n=\infty}\frac{\Delta_n^{(r)}}{\Delta_n}=a_{\frac{r+1}{2}}\qquad (r=1,\ 3,\cdots\cdots,\ 2n-1),$$
$$=b_{\frac{r}{2}}\qquad (r=2,\ 4,\cdots\cdots,\ 2n),$$

the limits (the F o u r i e r *constants) being independent of* a *and* b[1].

Consider the identity due to Prof. I. S c h u r[2]:

$$(2)\quad \begin{vmatrix} \int_a^b \varphi_0(x)\,\psi_0(x)\,dx & \int_a^b \varphi_0(x)\,\psi_1(x)\,dx\cdots\cdots & \int_a^b \varphi_0(x)\,\psi_p(x)\,dx \\ \int_a^b \varphi_1(x)\,\psi_0(x)\,dx & \int_a^b \varphi_1(x)\,\psi_1(x)\,dx\cdots\cdots & \int_a^b \varphi_1(x)\,\psi_p(x)\,dx \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \int_a^b \varphi_p(x)\,\psi_0(x)\,dx & \int_a^b \varphi_p(x)\,\psi_1(x)\,dx\cdots\cdots & \int_a^b \varphi_p(x)\,\psi_p(x)\,dx \end{vmatrix}$$

$$=\frac{1}{(p+1)!}\underbrace{\int_a^b\int_a^b\cdots\int_a^b}_{(p+1)} dx_0\,dx_1\cdots\cdots dx_p$$

$$\cdot\begin{vmatrix} \varphi_0(x_0) & \varphi_0(x_1)\cdots\cdots & \varphi_0(x_p) \\ \varphi_1(x_0) & \varphi_1(x_1)\cdots\cdots & \varphi_1(x_p) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \varphi_p(x_0) & \varphi_p(x_1)\cdots\cdots & \varphi_p(x_p) \end{vmatrix}\cdot\begin{vmatrix} \psi_0(x_0) & \psi_0(x_1)\cdots\cdots & \psi_0(x_p) \\ \psi_1(x_0) & \psi_1(x_1)\cdots\cdots & \psi_1(x_p) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \psi_p(x_0) & \psi_p(x_1)\cdots\cdots & \psi_p(x_p) \end{vmatrix}.$$

---

(1) This belongs to the case where the so-called "principe des réduites" is valid. See F. R i e s z, Les systèmes d'équations linéaires à une infinité d'inconnues (1913), p. 7.

(2) I. S c h u r, Zur Theorie der linearen homogenen Integralgleichungen, Math. Ann. 67 (1909), p. 319; R i c h a r d s o n—W. A. H u r w i t z, Note on determinants whose terms are certain integrals, Bull. Amer. Math. Soc., 16 (1909-10); L a n d s b e r g, Theorie der Elementarteiler linearer Integralgleichungen, Math. Ann. 69 (1910), p. 231. For an interesting application of this identity, see F u j i w a r a, Ein von B r u n n vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der S c h w a r z schen und der T c h e b y c h e f fschen Ungleichungen für bestimmte Integrale, Tôhoku Math. Journal, 13 (1918), p. 231.

If we put

$$\begin{aligned}
&\varphi_0(x)=\psi_0(x)=1,\\
&\varphi_1(x)=\psi_1(x)=\cos x,\\
&\varphi_2(x)=\psi_2(x)=\sin x,\\
&\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots\\
&\varphi_{2n-1}(x)=\psi_{2n-1}(x)=\cos nx,\\
&\varphi_{2n}(x) \quad =\psi_{2n}(x)=\sin nx;
\end{aligned}$$

and

$$D_n(x_0, x_1, \cdots\cdots, x_{2n})=\begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ \cos x_0 & \cos x_1 & \cdots\cdots & \cos x_{2n} \\ \sin x_0 & \sin x_1 & \cdots\cdots & \sin x_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cos nx_0 & \cos nx_1 & \cdots\cdots & \cos nx_{2n} \\ \sin nx_0 & \sin nx_1 & \cdots\cdots & \sin nx_{2n} \end{vmatrix},$$

then (2) becomes

$$(3) \quad \Delta_n=\frac{1}{(2n+1)!}\underbrace{\int_a^b\int_a^b\cdots\int_a^b}_{(2n+1)}[D_n(x_0, x_1, \cdots\cdots, x_{2n})]^2 dx_0\, dx_1 \cdots\cdots dx_{2n}.$$

It is seen that $D_n(x_0, x_1, \cdots\cdots, x_{2n})$ is not zero except

$$x_i=x_k, \qquad (i, k=0, 1, 2, \cdots\cdots, 2n),$$

in virtue of the identity

$$D_n(x_0,\ x_1, \cdots\cdots,\ x_{2n})=2^{2n^2}\prod \sin\frac{1}{2}(x_p-x_q)\ (^1),$$

where p, q are all duads from $0, 1, \cdots\cdots, 2n$, $(p>q)$.

Again if we put

$$\begin{aligned}
&\varphi_0(x)=1, \qquad \psi_0(x)=f(x),\\
&\varphi_1(x)=\psi_1(x)=\cos x,\\
&\varphi_2(x)=\psi_2(x)=\sin x,\\
&\cdots\cdots \quad \cdots\cdots \quad \cdots\cdots\\
&\varphi_{2n-1}(x)=\psi_{2n-1}(x)=\cos nx,\\
&\varphi_{2n}(x)=\psi_{2n}(x)=\sin nx,
\end{aligned}$$

(1) Scott and Mathews, Theory of determinants (2, ed.. 1904), p. 272.

(2) becomes

$$(4)\quad \Delta_n^{(0)}=\frac{1}{(2n+1)!}\underbrace{\int_a^b\int_a^b\cdots\int_a^b}_{(2n+1)} dx_0\,dx_1\cdots\cdots dx_{2n}\,D_n(x_0,x_1,\cdots\cdots,x_{2n})$$

$$\cdot\begin{vmatrix} f(x_0) & f(x_1) & \cdots\cdots & f(x_{2n}) \\ \cos x_0 & \cos x_1 & \cdots\cdots & \cos x_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sin nx_0 & \sin nx_1 & \cdots\cdots & \sin nx_{2n} \end{vmatrix}$$

Since the series (1) converges uniformly to $f(x)$ at every point of the interval $(0\leqq x\leqq 2\pi)$, if we put

$$f(x)=a_0+(a_1\cos x+b_1\sin x+\cdots\cdots+a_n\cos nx+b_n\sin nx)+R_n(x),$$

then, corresponding to any positive number ε, it may be possible to find a positive integer N which is independent of x $(a\leqq x\leqq b)$ and is such that

$$|R_n(x)|<\varepsilon,\qquad n>N.$$

Also we have the identities

$$\begin{vmatrix} f(x_0) & f(x_1) & \cdots & f(x_{2n}) \\ \cos x_0 & \cos x_1 & \cdots & \cos x_{2n} \\ \cdots\cdots & \cdots & \cdots & \cdots\cdots \\ \sin nx_0 & \sin nx_1 & \cdots & \sin nx_{2n} \end{vmatrix}=\begin{vmatrix} a_0+R_n(x_0) & a_0+R_n(x_1) & \cdots\cdots & a_0+R_n(x_{2n}) \\ \cos x_0 & \cos x_1 & \cdots\cdots & \cos x_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \sin nx_0 & \sin nx_1 & \cdots\cdots & \sin nx_{2n} \end{vmatrix}$$

$$=a_0D_n(x_0,x_1,\cdots,x_{2n})+\begin{vmatrix} R_n(x_0) & R_n(x_1) & \cdots & R_n(x_{2n}) \\ \cos x_0 & \cos x_1 & \cdots & \cos x_{2n} \\ \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ \sin nx_0 & \sin nx_1 & \cdots & \sin nx_{2n} \end{vmatrix}.$$

Now let $D_n^{(r)}(x_0,x_1,\cdots\cdots,x_{2n})$ be the determinant obtained from $D_n(x_0,x_1,\cdots\cdots,x_{2n})$ by replacing the elements of the $(r+1)$-th row by the elements

$$R_n(x_0),\ R_n(x_1),\cdots\cdots,\ R_n(x_{2n}).$$

Then (4) becomes

$$\Delta_n^{(0)}=\frac{a_0}{(2n+1)!}\int_a^b\int_a^b\cdots\int_a^b[D_n(x_0,x_1,\cdots,x_{2n})]^2dx_0\,dx_1\cdots dx_{2n}\text{ (1)}$$

(1) The number of the signs of integration in these expressions is $2n+1$.

$$+\frac{1}{(2n+1)!}\int_a^b\int_a^b\cdots\int_a^b D_n(x_0, x_1, \cdots, x_{2n})\cdot D_n^{(0)}(x_0, x_1, \cdots, x_{2n})\, dx_0 dx_1\cdots dx_{2n};$$

so that we have from (2)

(5) $\Delta_n^{(0)}/\Delta_n$

$$=a_0+\frac{\int_a^b\int_a^b\cdots\int_a^b D_n(x_0, x_1, \cdots, x_{2n})\cdot D_n^{(0)}(x_0, x_1, \cdots, x_{2n}) dx_0 dx_1\cdots dx_{2n}}{\int_a^b\int_a^b\cdots\int_a^b [D_n(x_0, x_1, \cdots, x_{2n})]^2\, dx_0 dx_1\cdots dx_{2n}}.$$

Similarly we obtain

(5′) $\Delta_n^{(r)}/\Delta_n$

$$=a_{\frac{r+1}{2}}+\frac{\int_a^b\int_a^b\cdots\int_a^b D_n(x_0, x_1, \cdots, x_{2n})\cdot D_n^{(r)}(x_0, x_1, \cdots, x_{2n})\, dx_0 dx_1\cdots dx_{2n}}{\int_a^b\int_a^b\cdots\int_a^b [D_n(x_0, x_1, \cdots, x_{2n})]^2\, dx_0\, dx_1 \cdots dx_{2n}}$$

$$(r=1, 3, \cdots, 2n-1),$$

$$=b_{\frac{r}{2}}+\frac{\int_a^b\int_a^b\cdots\int_a^b D_n(x_0, x_1, \cdots, x_{2n})\cdot D_n^{(r)}(x_0, x_1, \cdots, x_{2n})\, dx_0 dx_1\cdots dx_{2n}}{\int_a^b\int_a^b\cdots\int_a^b [D_n(x_0, x_1, \cdots, x_{2n})]^2 dx_0 dx_1\cdots dx_{2n}}$$

$$(r=2, 4, \cdots, 2n).$$

Now let $(\xi_0, \xi_1, \cdots, \xi_{2n})$ be the rectangular point coordinates in space of $2n+1$ dimensions, and consider the $2n+1$ planes

$$\xi_0+\xi_1 \cos x_0+\xi_2 \sin x_0+\cdots\cdots+\xi_{2n} \sin nx_0=R_n(x_0),$$
$$\xi_0+\xi_1 \cos x_1+\xi_2 \sin x_1+\cdots\cdots+\xi_{2n} \sin nx_1=R_n(x_1),$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots \qquad \cdots\cdots\cdots$$
$$\xi_0+\xi_1 \cos x_{2n}+\xi_2 \sin x_{2n}+\cdots\cdots+\xi_{2n} \sin nx_{2n}=R_n(x_{2n}).$$

The angle $\theta_n(x_i, x_k)$ between the i-th and k-th planes is given by

$$\cos\theta_n(x_i, x_k)=\frac{1+\sum_{p=1}^{n}(\cos px_i \cos px_k+\sin px_i \sin px_k)}{\sqrt{1+\sum_{p=1}^{n}(\cos^2 px_i+\sin^2 px_i)}\sqrt{1+\sum_{p=1}^{n}(\cos^2 px_k+\sin^2 px_k)}}$$

$$=\left\{1+\sum_{p=1}^{n}\cos p\,(x_i-x_k)\right\}\div(1+n)$$

$$=\frac{\cos\frac{n}{2}(x_i-x_k)\cdot\sin\frac{n+1}{2}(x_i-x_k)}{(1+n)\sin\frac{1}{2}(x_i-x_k)}.$$

Hence $\cos\theta_n(x_i, x_k)$ converges uniformly to zero, that is, the angle $\theta_n(x_i, x_k)$ converges uniformly to $\frac{\pi}{2}$, as n tends to infinity, when x_i, x_k lie in the interval (a, b) except $|x_i-x_k|<\delta$. Next the distance of i-th plane from the origin of coordinates is

$$\frac{R_n(x_i)}{\sqrt{1+n}}$$

which converges uniformly to zero as n tends to infinity. Consequently the common point of the $2n+1$ planes tends uniformly to the origin as n tends to infinity, except

$$|x_i-x_k|<\delta \quad (i, k=0, 1, 2, \cdots\cdots).$$

But the coordinates of the common point of these planes are

$$\xi_r(x_0, x_1, \cdots\cdots, x_{2n})=\frac{D_n^{(r)}(x_0, x_1, \cdots\cdots, x_{2n})}{D_n(x_0, x_1, \cdots\cdots, x_{2n})}, \quad (r=0, 1, 2, \cdots\cdots, 2n).$$

Hence if $x_0, x_1, \cdots\cdots, x_{2n}$ lie in the interval (a, b), excluding the domain defined by

$$|x_i-x_k|<\delta \quad (i, k=0, 1, 2, \cdots\cdots, 2n),$$

then, corresponding to any positive numbers $\varepsilon_r\ (r=0, 1, 2, \cdots\cdots, 2n)$, it may be possible to find a positive integer N, which is independent of $x_0, x_1, \cdots\cdots, x_{2n}$ and is such that

$$|\xi_r(x_0, x_1, \cdots\cdots, x_{2n})|<\varepsilon_r, \qquad n>N.$$

But

$$\frac{\int_a^b\cdots\int_a^b D_n(x_0, \cdots, x_{2n})\, D_n^{(r)}(x_0, \cdots, x_{2n})\, dx_0\cdots dx_{2n}}{\int_a^b\cdots\int_a^b [D_n(x_0, \cdots, x_{2n})]^2 dx_0\cdots dx_{2n}}$$

$$=\frac{\int_a^{*b}\cdots\int_a^{*b}\xi_r(x_0, \cdots, x_{2n})\cdot[D_n(x_0, \cdots, x_{2n})]^2 dx_0\cdots dx_{2n}}{\int_a^b\cdots\int_a^b [D_n(x_0, \cdots, x_{2n})]^2 dx_0\cdots dx_{2n}}$$

$$+\frac{\int_{*a}^{b}\cdots\int_{*a}^{b} D_n(x_0,\cdots,x_{2n})\,D_n^{(r)}(x_0,\cdots,x_{2n})\,dx_0\cdots dx_{2n}}{\int_a^b\cdots\int_a^b[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}},$$

where $\int^*$ denotes the integration excluding the domains $|x_i-x_k|<\delta$, $(i, k=0, 1,\cdots, 2n)$, and $\int_*$ that over these domains only.

Now

$$\left|\frac{\int_a^{*b}\cdots\int_a^{*b}\xi_r(x_0,\cdots,x_{2n})\cdot[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}}{\int_a^b\cdots\int_a^b[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}}\right|$$

$$<\frac{\int_a^{*b}\cdots\int_a^{*b}|\xi_r(x_0,\cdots,x_{2n})|\cdot[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}}{\int_a^{*b}\cdots\int_a^{*b}[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}}$$

$$<\varepsilon_r\cdot\frac{\int_a^{*b}\cdots\int_a^{*b}[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}}{\int_a^{*b}\cdots\int_a^{*b}[D_n(x_0,\cdots,x_{2n})]^2dx_0\cdots dx_{2n}}$$

$$=\varepsilon_r,\qquad n>N,\qquad (r=0,\,1,\,2,\cdots\cdots,\,2n).$$

On the other hand we may put

$$D_n^{(0)}(x_0,\cdots,x_{2n})=A_0\cdot R_n(x_0)+A_1\cdot R_n(x_1)+\cdots+A_{2n}\cdot R_n(x_{2n}),$$

where A_p is the algebraic complement of $R_n(x_p)$ in the determinant $D_n^{(0)}(x_0,\cdots x_{2n})$, i.e.,

$$A_0=\begin{vmatrix}\cos x_1 & \cos x_2 & \cdots\cdots & \cos x_{2n}\\ \sin x_1 & \sin x_2 & \cdots\cdots & \sin x_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ \cos nx_1 & \cos nx_2 & \cdots\cdots & \cos nx_{2n}\\ \sin nx_1 & \sin nx_2 & \cdots\cdots & \sin nx_{2n}\end{vmatrix},\qquad\cdots\cdots\cdots.$$

But we have

$$|A_0|=2^{2n^2-2n+1}\cdot\Pi\,\sin\frac{1}{2}(x_i-x_k)\;\cdot|S|\,(^1),$$

(1) Scott and Mathews, loc. cit., p. 272.

where

$$S=\Sigma \cos \frac{1}{2}\,(x_{i_1}+x_{i_2}+\cdots+x_{i_n}-x_{i_{n+1}}-\cdots-x_{i_{2n}})$$

is formed by dividing the $2n$ angles $x_1, x_2, \cdots\cdots, x_{2n}$ into two sets of n angles in all possible ways and taking the cosine of half the difference of the sums of these sets; so that there exists the inequality

$$|S|<(2n)!/(n!)^2,$$

which becomes, by virtue of Stirling's formula,

$$|S|<\frac{1}{\sqrt{n\pi}}\,2^{2n}+\lambda_n, \qquad (\lim_{n=\infty} \lambda_n=0).$$

If η be any fixed positive number smaller than 1, we may take δ such that

$$\left|\sin \frac{1}{2}\,(x_i-x_k)\right|<\eta \quad \text{for} \quad |x_i-x_k|<\delta, \quad (i, k=0, 1, 2, \cdots\cdots, 2n)$$

Hence

$$|A_0|<2^{2n^2+1}\left(\frac{1}{\sqrt{n\pi}}+\frac{\lambda_n}{2^{2n}}\right)\eta^n=2^{2n^2}\,\eta^n\,\lambda_n' \quad \text{for } |x_i-x_k|<\delta,$$

where

$$\lim_{n=\infty} \lambda_n'=0.$$

By similar ways we have

$$|A_p|<2^{2n^2}\,\eta^n\,\lambda_n' \quad (p=1, 2, \cdots, 2n) \quad \text{for } |x_i-x_k|<\delta.$$

Consequently

$$|D_n^{(0)}(x_0, \cdots, x_{2n})|<\sum_{p=0}^{2n}|A_p|\cdot|R_n(x_p)|<(2n+1)2^{2n^2}\,\eta^n\,\lambda_n'\,\varepsilon,$$
$$\text{for } |x_i-x_k|<\delta, \quad n>N.$$

Since

$$\lim_{n=\infty} (2n+1)\,\eta^n=0,$$

it may be possible to find the positive integer N' such that

$$\varepsilon\,\lambda_n'\cdot(2n+1)\,\eta^n<\varepsilon', \quad n>N'>N$$

corresponding to any positive number ε'; whence we obtain

$$|D_n^{(0)}(x_0, \cdots, x_{2n})|<2^{2n^2}\,\varepsilon', \quad n>N', \; |x_i-x_k|<\delta.$$

Similarly

$$|D_n^{(r)}(x_0,\cdots,x_{2n})|<2^{2n^2}\varepsilon', \quad n>N', \; |x_i-x_k|<\delta.$$
$$(r=1, 2, \cdots\cdots, 2n).$$

Therefore

$$\left|\frac{\int_{*a}^{b}\cdots\int_{*a}^{b} D_n(x_0,\cdots,x_{2n})\, D_n^{(r)}(x_0,\cdots,x_{2n})\, dx_0\cdots dx_{2n}}{\int_a^b\cdots\int_a^b [D_n(x_0,\cdots,x_{2n})]^2\, dx_0\cdots dx_{2n}}\right|$$

$$=\frac{\left|\int_{*a}^{b}\cdots\int_{*a}^{b} \Pi \sin\frac{1}{2}(x_i-x_k)\cdot D_n^{(r)}(x_0,\cdots,x_{2n})\, dx_0\cdots dx_{2n}\right|}{2^{2n^2}\int_a^b\cdots\int_a^b \left[\Pi \sin\frac{1}{2}(x_i-x_k)\right]^2 dx_0\cdots dx_{2n}}$$

$$<\varepsilon' \div \int_a^b\cdots\int_a^b \left[\Pi \sin\frac{1}{2}(x_i-x_k)\right]^2 dx_0\cdots dx_{2n}, \quad n>N', \; (r=0, 1, 2,\cdots, 2n).$$

Thus we arrive at the identities

$$\lim_{n=\infty}\frac{\int_a^b\cdots\int_a^b D_n(x_0,\cdots,x_{2n})\, D_n^{(r)}(x_0,\cdots,x_{2n})\, dx_0\cdots dx_{2n}}{\int_a^b\cdots\int_a^b [D_n(x_0,\cdots,x_{2n})]^2\, dx_0\cdots dx_{2n}}=0,$$
$$(r=0, 1, 2,\cdots, 2n).$$

Consequently it follows from (5) and (5′) that

$$\lim_{n=\infty} \Delta_n^{(0)}/\Delta_n=a_0,$$
$$\lim_{n=\infty} \Delta_n^{(r)}/\Delta_n=a_{\frac{r+1}{2}} \qquad (r=1, 3,\cdots\cdots, 2n-1),$$
$$=b_{\frac{r}{2}} \qquad (r=2, 4,\cdots\cdots, 2n).$$

Lastly we remark that the method of proof may be applied to any function defined by the series, which is uniformly convergent in the interval $(0\leqq x\leqq 2\pi)$, of orthogonal functions $\varphi_n(x)$, such that

$$\int_0^1 \varphi_m(x)\, \varphi_n(x)\, dx=0 \qquad m\neq n$$
$$=1 \qquad m=n,$$

and have one of the forms

$$\varphi_n(x)=k\cos(n\pi x+k')+\frac{\omega(n, x)}{n},$$

$$\varphi_n(x)=k\cos(2n\pi x+k')+\frac{\omega(n, x)}{n},$$

$$\varphi_n(x)=k\cos\left(\frac{2n+1}{n}\pi x+k'\right)+\frac{\omega(n, x)}{n} \text{ (1)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots\cdots\cdots,$$

where k, k' are constants and $|\omega(n, x)|$ is smaller than a finite number independent of n and x.

In such a case, if we take any two numbers a and b within the interval $(0, 1)$, the theorem similar to the above holds good also. The series of the Sturm-Liouville functions, that occurring in the theory of cooling of a sphere, that in the theory of lateral vibration of a bar, etc. belong to this case(2).

Ikeda near Ôsaka, April 1918.

---

(1) In these three forms we may replace the cosines by the sines respectively.

(2) Kneser, loc. cit.; Juretzka, loc. cit.; Ogura, Note on the representation of an arbitrary function in mathematical physics, Tôhoku Math. Journal, 1 (1911-12), p. 120.

Bemerkungen über die Beziehung zwischen Erfahrung und Geometrie,

VON

HASIME TANABE in Sendai.

1. Mancher Mathematiker meint, dass die Geometrie nicht eine apriorische, sondern eine empirische Wissenschaft sei. Gauss hat schon in seinem Brief an Bessel, 1829, gesagt: „Nach meiner innigsten Überzeugung hat die Raumlehre zu unserem Wissen der selbstverständlichen Wahrheiten eine ganz andere Stellung, als die reine Grössenlehre; es geht unserer Kenntnis von jener durchaus diejenige vollständige Überzeugung von ihrer Notwendigkeit (also auch von ihrer absoluten Wahrheit) ab, welche der letzteren eigen ist; wir müssen in Demut zugeben, dass wenn die Zahl bloss unseres Geistes Produkt ist, der Raum auch ausser Geiste eine Realität hat, der wir *a priori* ihre Gesetze nicht vollständig vorschreiben können.“ Es ist die Ansicht der meisten Mathematiker, dass die Geometrie nicht bloss psycho-genetisch, sondern auch logisch von der Erfahrung abhängig sei. Im folgenden möchte ich darüber einige kritische Bemerkungen zu machen versuchen.

Zuerst muss ich den Sinn des Begriffs „Erfahrung" transzendentalphilosophisch präzis bestimmen. Man meint gemeinhin von dem naivrealistischen Standpunkte aus, die Erfahrung ist das in unserem Bewusstsein entstandene Abbild der davon unabhängig uns gegenüberliegenden Welt. Ein Verdienst Kants war es, dass er solche erkenntnistheoretische Abbildtheorie gänzlich umgestürzt hat, indem er zeigte, dass die Erfahrung erst dadurch entsteht, dass das Denken nach den apriorischen Formen (oder sogenannten Kategorien) die gegebenen Empfindungsdaten ordnet. Diese Ansicht heisst aprioristische Konstruktionstheorie. Von dem von uns nach Kants Vorgang angenommenen kritischen Standpunkte ist die Erfahrung kein Abbild der Welt von Dingen an sich, sondern ein Konstruktionsprodukt des Subjektes. Der Raum ist eine von den für solche Konstruktion vorausgesetzten Formen. Er ist kein empirischer, sondern ein apriorischer Begriff. Darum muss die Geometrie als Lehre vom Raum auch apriorisch sein, wenn ihre Axiome alle notwendige Bestimmungen des Raumes als Erfahrungsform darstellen. Wenn

dagegen die Axiome mehr als diese erfordern, so kommt die Frage hervor, ob die darauf gegründete Geometrie empirisch sein müsse, oder noch in irgend einem Sinne apriorisch sein könne. Und wenn das letztere der Fall ist, fragt es sich noch weiter, in welchem Sinne die Geometrie dann für apriorisch gehalten werden darf.

2. Der Raum als Erfahrungsform könnte als die Ordnungsform der gleichzeitig, nebeneinander, aussereinander, beisammen existierenden Gegenstände gedeutet werden. Aber unser Denken kann nicht gleichzeitig viele Objekte unterscheidend setzen; es muss sie nacheinander sukzessiv setzen. Nur im Gegensatz zu der Zeit beharren im Raume alle gesetzten Objekte. In dieser Hinsicht ist der letztere die Form der Allheit, während die erstere die der Vielheit ist. Die Totalität der nacheinander gesetzten Gegenstände bilden eine geradlinige Reihe. Und eine solche Reihe selbst kann auch als einheitliches Objekt zu einem Element einer neuen Reihe werden. So entsteht der Raum als Reihe der Reihen in derselben Weise, wie das vieldimensionale System der komplexen Zahlen. Es ist auch ersichtlich, dass jener in seiner konkreten Bildung ein n-dimensionales, stetiges, unendliches System der Elemente wie dieses (n, eine beliebige positive, ganze Zahl). Homogeneität und Isotropie kommen ihm ohne weiteres zu. Der Begriff von „Punkt" stellt einen einzelnen von einander nur ordnungsmässig zu unterscheidenden Gegenstand dar. Daher ist er einfach (dies drückt man so aus, der Punkt hat keine Grösse). Durch zwei Punkte wird eine Gerade bestimmt, und durch drei nicht in einer Gerade liegende Punkte eine Ebene, u. s. w., u. s. w. Wir bekommen derart ganz logisch einen vieldimensionalen, homogenen, isotropen, unendlichen, stetigen Raum. Solche Eigenschaften gehören ihm als apriorischer Form der Erfahrung zu. Sie sind daher alle apriorisch, nicht empirisch, in dem oben angegebenen Sinne.

Von diesem rein logischen Gesichtspunkte kann man keine besondere Zahl als Anzahl der Dimensionen den anderen vorziehen. Wir dürfen eine beliebige positive, ganze Zahl dafür annehmen. Nur braucht eine gewisse Zahl ein für allemal gewählt werden. Der Raum von irgend einer Anzahl der Dimensionen besteht als Gegenstand der Geometrie mit gleichem Recht. Nun nimmt die gewöhnliche Euklidische Geometrie den drei-dimensionalen Raum als Gegenstand an. Es fragt sich aber, durch welches Merkmal diese Geometrie solchen Raum auswählt. Mancher Mathematiker möge darauf so antworten, dass die Erfahrung dieser Auswahl eine Leitung gebe. Er ist der Meinung, dass der Erfahrungsraum drei Dimensionen hat, und dass die Euklidische Geometrie als

Lehre vom Erfahrungsraum den drei-dimensionalen Raum als Gegenstand annehmen muss. Der Euklidische Raum ist nach ihm Erfahrungsraum. Aber was bedeutet die Behauptung, dass der Euklidische Raum Erfahrungsraum sei? Wie ich oben vom kritischen Standpunkte aus gedeutet habe, muss der Sinn dieser Behauptung etwa das folgende sein. Der Euklidische Raum gibt die adäquate Ordnungsform der Erfahrung, das heisst, die durch Euklidische Geometrie gegebenen Bestimmungen sind notwendig und hinreichend für die räumliche Konstruktion der Erfahrung. Mehr als drei dimensionaler Raum kann ohne Zweifel die Erfahrung konstruieren; aber sie ist nicht notwendig, übrigens gibt es keinen Massstab für die Auswahl einer gewissen Anzahl. Für die einzig bestimmte Konstruktion der Erfahrung muss die für das Ziel hinreichende sowohl als notwendige Form angenommen werden. Der drei-dimensionale Raum ist der einzige, der diesem Erfordernis genügt. Er ist dadurch allein Erfahrungsraum, dass er der für die Konstruktion der Erfahrung hinreichende sowohl als notwendige Raum ist. Seine drei-Dimensionalität ist nicht ein empirisches Prädikat in demselben Sinne, wie die induktive Bestimmungen in der empirischen Wissenschaften. Die letztere begründen sich auf der Erfahrung, während die erstere eine die Erfahrung möglich machende Voraussetzung ist. Sofern kann man sagen, dass sie apriorisch ist.

3. Poincaré hat die folgenden als die wesentlichen Eigenschaften des geometrischen Raumes angegeben, nämlich, 1. Stetigkeit, 2. Unendlichkeit, 3. drei-Dimensionalität, 4. Homogeneität, 5. Isotropie(1).

Wir wissen nach dem Vorhergehenden, dass sie alle apriorische Bestimmungen sind. Unter den von Hilbert aufgestellten Axiomgruppen(2) werden diejenigen der Verknüpfung, Anordnung, Kongruenz und Stetigkeit zweifelsohne durch diese Eigenschaften begründet. Nur das sogenannte Axiom der Parallelen bekommt keine Begründung dadurch. Die Fragen bezüglich der Euklidischen und der nicht-Euklidischen Geometrie drehen sich wesentlich bloss um dieses Axiom. Nun ist die projektive Geometrie, wie man weiss, von diesem unabhängig. Sie gründet sich auf den Axiomen der Verknüpfung, Anordnung und Stetigkeit allein. Man beschäftigt sich in diesem Zweig der Geometrie bloss damit, wie die Grundgebilde durch sogenannte projektive Eigenschaften mit einander bezogen werden. Keine metrische Verhältnisse werden hier in

---

(1) Poincaré, Wissenschaft und Hypothese, S. 53—54.

(2) Hilbert, Grundlagen der Geometrie.

Betracht gezogen. Die reine, projektive Geometrie gilt unabhängig von dem Unterschied zwischen den Euklidischen und nicht-Euklidischen Räumen. So sieht man ohne weiteres, dass sie eine ganz apriorische Wissenschaft sein muss. Die drei-Dimensionalität des projektiven Raumes beeinträchtigt die Apriorität dieses Zweigs der Geometrie gar nicht, was man aus dem Vorangegangenen leicht erkennen wird. Dagegen in der metrischen Geometrie muss eine gewisse Bestimmung betreffs der Parallelen gegeben werden. Die Euklidische Richtung stellt das folgende Postulat als Axiom auf: es gibt in der durch eine beliebige Gerade und einen ausserhalb ihrer liegenden Punkt bestimmten Ebene nur eine Parallele, d. h. eine Gerade, die durch diesen Punkt läuft, und die Gerade nicht schneidet. Es ist die nicht-Euklidische Richtung, die dies Parallelaxiom verneint. Das Lobatschewsky-Bolyaische System sagt aus, dass es eine unendliche Anzahl der sich nicht schneidenden Geraden in ein und derselben Ebene. Es nennt Parallelen nur zwei von ihnen, welche durch den Abstand von dem Punkt zur Gerade in dem obigen Postulate eindeutig bestimmt werden. Und in diesem System ist die Summe der inneren Winkel eines ebenen Dreiecks kleiner als zwei Rechte; die Differenz ist dem Flächeninhalte des Dreiecks proportional. Zuletzt in dem Riemann-Helmholtzschen System gibt es keine Parallele; jedes Paar Geraden in derselben Ebene schneiden sich immer. Und in diesem System ist die Summe der Winkel eines ebenen Dreiecks stets grösser als zwei Rechte; der Überschuss ist auch diesmal dem Flächeninhalte des Dreiecks proportional.

Vom rein mathematischen Standpunkte aus betrachtet, gibt es keinen Wertunterschied unter drei soeben besprochenen Systemen. Sie stehen alle mit gleichem Recht. Man dürfte daran zweifeln, dass mehrere sich widersprechenden Bestimmungen von denjenigen Gegenständen nebeneinander gleichfalls wahr sind. Jedoch gibt es, näher betrachtet, hier keinen Widerspruch, denn in jedem von drei Systemen haben dieselben Namen verschiedene Bedeutungen; z. B. mit demselben Namen „Gerade“ benennt man ganz verschiedene Gegenstände. Solche Verschiedenheiten führen sich zu denen der sogenanten Raumkrümmungsmasse zurück. In jedem oben gekennzeichneten System hat der Raum einen verschiedenen Wert von Krümmungsmass, nämlich, der Euklidische Raum Null, während der Lobatschewsky-Bolyaische einen negativen, und der Riemann-Helmholtzsche einen positiven Wert hat. In jedem Raum von konstantem Krümmungsmass gilt das Kongruenzaxiom, welches jeder Metrik zugrunde liegt. Aber es kann

nicht logisch eindeutig bestimmt werden, welcher Wert des Krümmungsmasses angenommen werden muss. Drei verschiedene Systeme von Geometrien stehen mit gleichem Recht. Überdies ist es gezeigt, dass irgendein System in jedem andern partiell projiziert werden kann, und dass somit sie alle miteinander gleichzeitig stehen oder fallen müssen. Als reine Mathematik sind sie alle gleichermassen wahr.

4. Die Frage über die Auswahl unter dem Euklidischen und den nicht-Euklidischen Systemen tritt erst dann hervor, wenn man die Geometrie als Geometrie der Naturwissenschaften betrachtet. Man fragt nämlich danach, welches System der Geometrie der empirischen Erkenntnis der Naturwissenschaften adäquat ist. In dieser Hinsicht allein erhält die Euklidische Geometrie unter anderen Systemen Vorzug. Sie bekommt eine hervorragende Stellung nur aus dem Gesichtspunkt der angewandten Mathematik betrachtet. Was bedeutet es aber, dass die Euklidische Geometrie der empirischen Erkenntnis adäquat ist? Man meint gemeinhin, dass die Euklidische Geometrie den gegebenen Erfahrungsraum als Gegenstand ihrer Erkenntnis hat, und so ihn genau abbildet. Die Euklidische Geometrie soll mit den Erfahrungstatsachen übereinstimmen. Aber der Erfahrungsraum ist, von dem kritischen Standpunkte aus betrachtet, wie schon oben hervorgehoben, nicht etwa ein Rahmen der uns gegebenen Erfahrungswelt. Die räumlichen Bestimmungen einer Erfahrungstatsache sind uns im kritischen Sinne nicht gegeben. Man kann sie nicht direkt von ihr ablesen. Die Erfahrung schreibt der Geometrie eindeutige Bestimmungen nicht vor, wie bei den empirischen Naturgesetzen der Fall ist. Im Gegenteil entsteht die räumliche Bestimmung einer Erfahrungstatsache erst nach der Geometrie, und kann man auch nach der nicht-Euklidischen Geometrie eine Erfahrungstatsache sogut bestimmen, wie nach der Euklidischen. Man kann die eigentümliche Bedeutung der letzteren nicht durch die Übereinstimmung mit der Erfahrung begründen. So hält Poincaré für unmöglich, mit dem Empirismus in der Geometrie einen vernünftigen Sinn zu verbinden[1]. Das Euklidische Parallelaxiom kann nicht eine empirische Wahrheit sein.

Schon in der Anfangsperiode der Geschichte der nicht-Euklidischen Geometrie hat man es in den Sinn kommen lassen, dass man die Frage, welches System der Geometrie wahr ist, durch die astronomischen Beobachtungen entscheiden kann. Wenn die Lobatschewskysche Geometrie wahr, so sei die Parallaxe eines sehr entfernten Sternes positiv endlich;

---

(1) Op. cit., S. 81.

wenn die Riemannsche Geometrie wahr, so sei sie negativ endlich; und wenn letztlich die Euklidische Geometrie wahr sei, so werde sie Null sein. Aber diese Meinung ist, wie Poriscaré bemerkt, ganz grundlos. Wenn die Beobachtungen etwa einen endlichen Wert als Resultat gäbe, so würde doch der Astronom noch ungestört die Euklidische Geometrie als wahr annehmen dürfen, indem er die betreffenden physikalischen Gesetze entsprechend modifiziere. Alle Erscheinungen, welche im nicht-Euklidischen Raum möglich sind, sind auch so im Euklidischen Raum, und umgekehrt. Die Frage, welche Geometrie wahr ist, kann nicht durch die Erfahrung entschieden werden. Ja, eine derartige Frage selbst ist, wie Poincaré mit Recht sagt, gleichermassen abgeschmackt wie die folgende: „Gibt es Längen, welche man in Metern und Zentimetern angeben kann, aber welche man nicht in Klafter, Fuss und Zoll abmessen kann?"(1) Poincaré lässt keinen Raum für die Frage über die Wahrheit der Geometrie; nach ihm kann eine Geometrie bloss einfacher und bequemer als die anderen sein. Die geometrischen Axiome sind nicht empirische Tatsachen, sondern auf Übereinkommen beruhende Festsetzungen, an denen wir eine Wahlfreiheit besitzen(2). Nun ist diese pragmatische Ansicht Poincarés sicherlich eine sehr beachtenswerte. Seine kritische Schärfe gegen den unter Mathematikern sehr verbreiteten Empirismus macht ihm Ehre. Aber von unserem kritischen Standpunkte aus betrachtet, lässt sein Pragmatismus noch etwas weiter zu kritisieren bleiben. Ist man wohl berechtigt, mit ihm zu sagen, die geometrischen Axiome sind keine apriorischen Urteile? (3)

5. Ich habe schon oben gezeigt, dass die anderen Axiome als das der Parallelen alle *apriori* sind. Jetzt beschäftigt uns nur die Frage betreffs des Parallelaxioms. Zuerst ist es nicht zu bestreiten, dass das Parallelaxiom notwendig von dem logischen Bau des Raumes nicht ableitbar ist. Der rein logische Begriff von Raum enthält keine eindeutige Bestimmung betreffend das Krümmungsmass in sich. Man kann nicht vom rein logischen Standpunkt das Problem entscheiden. Wir müssen den Raum als Konstruktionsform der wirklichen Erfahrungswelt ansehen, und die Folgen der verschiedenen Systemen der Geometrien von diesem Gesichtspunkte näher ins Betracht ziehen.

Zuerst, wie ich bemerkt habe, ist die Differenz zwischen zwei Rechten und der Summe der inneren Winkel eines ebenen Dreiecks in dem

(1) Op. cit., S. 75.
(2) Op. cit.. S. 51.
(3) Op. cit., S. 51.

nicht-Euklidischen Raume dem Flächeninhalte des Dreiecks proportional ist. Daher ist es in diesem Raume unmöglich, zu einer gegebenen Figur eine ähnliche Figur in grösseren oder kleineren Dimensionen zu zeichnen. Man kann hier von der Ähnlichkeit der Figuren nicht reden. Wir können den inneren Bau eines materiellen Körpers in dem nicht-Euklidischen Raume nicht unabhängig von seiner Grösse denken. Die räumliche Grösse erhält eine absolute Bedeutung. In anderen Worten kann man auch sagen, dass der Raum auf die Eigenschaften eines darin gegebenen empirischen Gegenstandes wirkt. Ein derartiger Raum kann nicht als reine Konstruktionsform im vollkommensten Sinne angesehen werden.

Überdies gibt es keinen Massstab für die Bestimmung des Krümmungsmasses bei dem nicht-Euklidischen Raume. Rein mathematisch ist wohl irgend ein konstanter Wert dafür gleichfalls annehmbar, aber der Raum als Konstruktionsform der wirklichen Erfahrungswelt muss einen einzig bestimmten Wert als sein Krümmungsmass erhalten. Vom kritischen Standpunkt bedeutet die Existenz die Einzigkeit der Bestimmungen. Der nicht-Euklidische Raum, der solch eine Unbestimmtheit an sich trägt, vermag keine einzig bestimmte Konstruktion der Existenzwelt ermöglichen. Der Euklidische Raum allein, der mit seinem Krümmungsmass Null vollständig bestimmt ist, befriedigt dieses Erfordernis. Er ist von diesem Gesichtspunkt ein einzig möglicher Raum der wirklichen Erfahrungswelt.

Letztens, was den Riemanschen Raum allein betrifft, ist dieser Raum zwar unbegrenzt, aber nicht unendlich. Jede Gerade, welche von einem Punkt in irgend einer Richtung gezogen wird, kommt in diesem Raum wieder nach einer bestimmten Länge zu diesem Punkt zurück. Die Unendlichkeit im eigentlichen Sinne kann man hier nicht statuieren. Dies auch entzieht diesem Raum die Berechtigung für die Konstruktionsform der Erfahrung.

Diese Betrachtungen zeigen uns, dass der Euklidische Raum allein zu der Konstruktion der Erfahrungswelt fähig ist. Er ist die einzige Raumform der Erfahrung, die das Subjekt annehmem soll. Die Euklidische Geometrie ist nicht bloss die einfachste und bequemste, wie die Pragmatisten wie z. B. Poincaré behaupten, sondern auch die einzig mögliche Geometrie des Erfahrungsraumes. Wenn man die Möglichkeit der Erfahrung anerkennen will, soll man dieser Geometrie einen eigentümlichen Vorzug gewähren. In diesem Sinne ist sie apriorisch, denn sie sich nicht auf der Erfahrung begründet, sondern im Gegenteil die

Erfahrung selbst möglich macht. Dreidimensionalität und Parallelaxiom der Euklidischen Geometrie sind in der Tat keine logisch notwendige Bestimmungen des Raumes, aber gehören jedoch zu der unerlässlichen Eigenschaften der die Erfahrung möglich machenden Raumform. Soweit sind sie auch apriorische wie die übrigen axiomatischen Bestimmungen. Daraus möchte ich folgern, dass die Geometrie nicht eine empirische, sondern eine apriorische Wissenschaft ist. Pragmatismus sowie Empirismus muss hier wie sonst für Apriorismus Feld räumen.

Zum Schluss möchte ich den Herren Prof. Hayashi und Prof. Ogura meinen herzlichen Dank sagen.

On the Dajutu or the Arithmetic Series of Higher Orders as Studied by Wasanists,

by

KITIZI YANAGIHARA, Sendai.

The *Dajutu* 垜術 is an important chapter in Wasan or Japanese native mathematics, to which a great attention of the Wasanists was paid. In it are studied the properties of the figurate numbers, in their language *Suida* 衰垜, the power sum $1^r+2^r+3^r+\cdots\cdots+n^r$, i. e. *Hôda* 方垜, and allied subjects.

Generally speaking, the Dajutu is a theory of arithmetic series of higher orders, and every series has its own name according to the laws of formation of its terms. The description and in some cases the verification of the results obtaind by the Wasanists with regard to the properties of Suida and Hôda will be the main object of the present paper.

I. Suida 衰垜.

The Suida are the figurate numbers, and the Wasanists called the figurate numbers of the t-th rank $(t-2)$-*Jô Suida* 乘衰垜, and particularly

Keida 圭垜	when $t=2$,
Sankaku Suida 三角衰垜	when $t=3$,
Saijô Suida 再乘衰垜	when $t=4$.

This irregurality of nomenclature will be due to the peculiarity that the Wasanists used to call $x^2, x^3, x^4, x^5, \cdots\cdots$, Heihô 平方, Rippô 立方, Sanjô 三乘, Yojô 四乘,$\cdots\cdots$ respectively.

Several formulae on the Suida are found in the Wasanists' work, which I will describe and verify here.

Let us denote the numbers in the i-th row and k-th column in the following arrangement of the figurate numbers

$$\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & \cdots\cdots \\ 1 & 2 & 3 & 4 & 5 & \cdots\cdots \qquad \text{Keida, 圭垜} \end{array}$$

1	3	6	10	15	......	Sankaku Suida, 三角衰垜
1	4	10	20	35	......	Saijô Suida, 再乘衰垜
1	5	15	35	70	......	Sanjô Suida, 三乘衰垜
......						......

by $$a_{i,k}\ (i, k=1,\ 2,\ 3, \cdots\cdots).$$

Then the law of formation of the terms is given by

$$a_{r,1}+a_{r,2}+a_{r,3}+\cdots\cdots+a_{r,n}=a_{r+1,n},$$

and the Wasanists called this sum "the Seki 積 of the $(r-2)$–Jô Suida for Teisi 底子 n", that is the sum of the first n terms of the figurate numbers of the r-th rank.

Ryôhitu Matunaga 松永良弼, one of the ablest Wasanists flourished in the first half of the eighteenth century, gave in his manuscript book Sanpô Zenkyô 算法全經 the relation between $a_{n,r+1}$ and $a_{n,r+2}$ as

(1) $$(a_{n,2}+r)a_{n,r+1}/(r+1)=a_{n,r+2} \quad \text{for} \quad r>1,$$

while

$$a_{1,r}=a_{r,1}=1, \qquad (r=1,\ 2,\ 3, \cdots\cdots).$$

In the manuscript book Daseki Syûhô 垜積拾法, the 45-th volume of Sekisan Kwanden 關算完傳 written by Hoyû Toita 戸板保佑 about 150 years ago we find the well known formula

(2) $$a_{r+1,n}=n(n+1)(n+2)\cdots\cdots(n+r-1)/r! \qquad (r>0)$$

The Wasanists called the denominator $r!$ in (2) Yakuhô 約法 (divisor), and the numerator

(2′) $$n(n+1)(n+2)\cdots\cdots(n+r-1)$$

the Suida Siki 衰垜式 of the r-th rank, the first three of which are

$$n^2+n \qquad \text{for} \qquad a_{2,n},$$
$$n^3+3n^2+2n \qquad \text{for} \qquad a_{3,n},$$
$$n^4+6n^3+11n^2+6n \qquad \text{for} \qquad a_{4,n}.$$

We can by actual calculation construct the following table of the coefficients in these expressions.

rank	divisor	coefficients
1	1	1 0
2	2	1 1 0

3	6	1	3	2	0			
4	24	1	6	11	6	0		
5	120	1	10	35	50	24	0	
6	720	1	15	85	225	274	120	0
......	......	......						

If we denote these coefficients in the i-th row and k-th column by

$$b_{i,k} \qquad (i, k=1, 2, 3, \cdots\cdots),$$

then we find in Matunaga's Sanpô Zenkyô, the relation

(3) $$b_{r,k}=(r-1).\, b_{r-1,k-1}+b_{r-1,k}.$$

This can also be found in Anmei Aida's 會田安明 (1747–1817) Sanpô Dajutu 算法垜術 vol. 1, in a modified form.

The method of finding the divisor d_r of the figurate numbers of the r-th rank, is given in Sanpô Zenkyô as

(4) $$d_r=r.\, b_{r,r} \qquad (r=1, 2, 3, \cdots\cdots);$$

that is to say, the divisor corresponding to the Suida Siki of the r-th rank is the product of the rank r and the last effective coefficient in the Suida Siki in consideration.

Besides this, Matunaga noticed that

(5) $$d_r=b_{r,1}+b_{r,2}+\cdots\cdots+b_{r,r};$$

that is to say, d_r is equal to the sum of all the coefficients of the powers of n in the Suida Siki of the r-th rank.

The validity of (1) is evident from (2). Let us now proceed to see if the formulæ (3), (4) and (5) are true or not.

First let us show that (3) is true. The Suida Siki (2′) of the r-th rank is written

$$b_{r,1}\, n^r+b_{r,2}\, n^{r-1}+b_{r,3}\, n^{r-2}+\cdots\cdots+b_{r,k}\, n^{r+1-k}+b_{r,k+1}\, n^{r-k}+\cdots\cdots+b_{r,r}\, n.$$

But since from (2′) the Suida Siki of the $(r+1)$-th rank is $n+r$ times that of the r-th rank, the Suida Siki of the fomer is

$$b_{r+1,1}\, n^{r+1}+(r.\, b_{r,1}+b_{r,2})n^r+(r.\, b_{r,2}+b_{r,3})n^{r-1}+\cdots\cdots$$
$$+(r.b_{r,k}+b_{r,k+1})n^{r-k+1}+\cdots\cdots\cdots\cdots.$$

Hence we get

$$b_{r+1,k+1}=r.\, b_{r,k}+b_{r,k+1},$$

what is the same thing as (3).

Next to show that (4) is true, we see from (2) that the divisor of

the Suida Siki of the r-th rank is $r!$, while the coefficient of the first power of n, i.e. $b_{r,r,}$ in the expansion of

$$n(n+1)(n+2)\cdots\cdots(n+r-1)$$

is evidently $(r-1)!$, so that $r.\,b_{r,r}=r!$.

Hence follows at once the relation (4).

Finally, if we put $n=1$, we get

$$n(n+1)(n+2)\cdots\cdots(n+r-1)=r!,$$

whence

$$b_{r,1}+b_{r,2}+b_{r,3}+\cdots\cdots+b_{r,r}=r!,$$

so that

$$d_r=b_{r,1}+b_{r,2}+\cdots\cdots+b_{r,r}.$$

II. Gensuida 減衰垜.

The following are called the Gensuida.

0	1	2	3	4	5	6……	Kei Genda	圭 減 垜,
0	0	1	3	6	10	15……	Sankaku Gensuida	三角減衰垜,
0	0	0	1	4	10	20……	Saijô Gensuida	再乘減衰垜,
0	0	0	0	1	5	15……	Sanjô Gensuida	三乘減衰垜,
……	……	……	……	……	……	……	……………	

If we denote these numbers by $c_{i,k}$, we get

$$\begin{aligned} c_{r,n}&=a_{r+1,\,n-r} \qquad \text{for} \qquad r<n,\\ &=0 \qquad\qquad\quad \text{for} \qquad r\geqq n,\end{aligned}$$

and the sum

$$c_{r,1}+c_{r,2}+c_{r,3}+\cdots\cdots+c_{r,n}\,(=C_{r,n}\ \text{say})$$

was called by the Wasanists "the sum of $(r-1)$-Jô Gensuida for Teisi n".

Daseki Syûhô contains the statement equivalent to

$$C_{r,n}=\frac{n(n-1)(n-2)(n-3)\cdots\cdots(n-r)}{(r+1)!}$$

which can at once be verified in the following way.

$$\begin{aligned} C_{r,n}&=c_{r,1}+c_{r,2}+\cdots\cdots+c_{r,r}+c_{r,r+1}+\cdots\cdots+c_{r,n}\\ &=c_{r,r+1}+c_{r,r+2}+\cdots\cdots+c_{r,n}\\ &=a_{r+1,1}+a_{r+2,2}+\cdots\cdots+a_{r+1,n-r}\\ &=a_{r+2,\,n-r}\end{aligned}$$

$$=\frac{(n-r)(n-r+1)\cdots\cdots(n-2)(n-1)n}{(r+1)!}.$$

III. Kirei Suida 奇零衰梁.

The Kirei Suida are the following.

1	3	5	7	9	11……	Kirei Keida	奇零圭梁,
1	4	9	16	25	36……	Kirei Sankaku Suida	奇零三角衰梁,
1	5	14	30	55	91……	Kirei Saijô Suida	奇零再乘衰梁,
1	6	20	50	105	196……	Kirei Sanjô Suida	奇零三乘衰梁.
…………						…………	

If one of these numbers be denoted by $p_{i,k}$, we get

$$p_{1,k}=2k-1,$$

$$p_{i,1}+p_{i,2}+p_{i,3}+\cdots\cdots+p_{i,n}=p_{i+1,n}.$$

and the sum in the last equality was called by the Wasanists "the sum of Kirei $(r-1)$–Jô Suida for Teisi $2n$".

The following is the law of formation of $p_{r+1,n}$, found in Daseki Syûhô.

If we put

$$M=(2n-1+1)(2n-1+3)(2n-1+5)\cdots\cdots(2n-1+2\,\overline{r-1}+1),$$

$$N=4.\ 6.\ 8\cdots\cdots(2+2r),$$

then

$$p_{r+1,n}=M(2+2r)\ /\ N$$

$$=\frac{2n(2n+2)(2n+4)\cdots\cdots(2n+2.\,\overline{r-1})}{4.\ 6.\ 8\cdots\cdots(2+2r)}(2n-1+r).$$

In Daseki Syûhô this relation is rendered easy to remember by the following scheme. To find the sum $p_{r,1}+p_{r,2}+\cdots\cdots+p_{r,n}$, write n numbers $1, 3, 5,\cdots\cdots$, to each of which is added Teisi $2n-1$, and then write n numbers $2, 4, 6,\cdots\cdots$, to each of which is added Teisi $2n-1$. Then the continued product of all the numbers in the left column multiplied by the last one $2+2r+2n-1$ in the right column when divided by the product of n numbers $4, 6, 8,\cdots\cdots$,

$1+2n-1$	$2+2n-1$
$3+2n-1$	$4+2n-1$
$5+2n-1$	$6+2n-1$
…………	…………
$2r-1+2n-1$	$2+2r+2n-1$

gives the required sum.

Anmei Aida observed, in his Sanpô Dajutu, that

$$1+3+5+7+\cdots\cdots+(2n-1)=n^2.$$

He also considered the two partial series of the Kireisuida.

$$1,\quad 5,\quad 9,\quad 13,\quad 17,\quad 21,\cdots\cdots$$

$$1,\quad 7,\quad 13,\quad 19,\quad 25,\quad 31,\cdots\cdots.$$

For the first, he proceeded as follows.

	1	2	3	4	5	6	7	......
terms	1	5	9	13	17	21	25	......
sums	1	6	15	28	45	66	91	......

From this we see that

$$1=1+4\times 0,$$
$$6=2+4\times 1,$$
$$15=3+4\times 3,$$
$$28=4+4\times 6,$$
$$45=5+4\times 10,$$
$$66=6+4\times 15,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Hence the sum

$$1+5+9+\cdots\cdots\text{to the } n\text{-th term}$$

is equal to $n+4.\, a_{2,\, n-1}.$

For the second,

	1	2	3	4	5	6	......
terms	1	7	13	19	25	31	......
sums	1	8	21	40	65	96	......

From this we see that

$$1=1+6\times 0,$$
$$8=2+6\times 1,$$
$$21=3+6\times 3,$$

$$40=4+6\times 6,$$
$$65=5+6\times 10,$$
$$\cdots\cdots\cdots\cdots$$

Hence the sum

$1+7+13\cdots\cdots$ to the n-th term is equal to $n+6a_{2,\,n-1}$.

IV. Gûrei Suida 偶零衰梁.

2	4	6	8······	Gûrei Keida	偶 零 圭 垜,
2	6	12	20······	Gûrei Sankaku Suida	偶零三角衰垜,
2	8	20	40······	Gûrei Saijô Suida	偶零再乘衰垜,
2	10	30	70······	Gûrei Sanjô Suida	偶零三乘衰垜,
······				··········	

If we denote these numbers by $q_{l,k}$, then

$$q_{1,k}=2k,$$

$$q_{r,1}+q_{r,2}+q_{r,3}+\cdots\cdots+q_{r,n}=q_{r+1,\,n} \qquad (r>1);$$

this sum was called "the sum of Gûrei Suida for Teisi $2n$".

Daseki Syûhô gives the following relation which can also be found in Sanpô Zenkyô.

$$q_{r+1,\,n}=\frac{(2n+2)(2n+4)\cdots\cdots(2n+2r)}{4.\,6.\,8\cdots\cdots(2n+2r)}2n.$$

V. Hôda 方垜.

The Hôda is the series of powers of natural numbers. The series

$$1^k,\ 2^k,\ 3^k,\cdots\cdots n^k,\cdots\cdots,$$

k being a positive integer, was called "$(k-1)$–Jô Hôda", and the sum

$$S_k(n)\equiv 1^k+2^k+3^k+\cdots\cdots+n^k,$$

"the sum of $(k-1)$–Jô Hôda for Teisi n".

When $k=1$, it is nothing but the Keida, and when $k=2,\,3$, it was specially called Heihôda 平方垜, Rippôda 立方垜 respectively.

The formulae

$$S_1(n)=\frac{1}{2}n(n+1)=\frac{1}{2}(n^2+n),$$

$$S_2(n)=\frac{1}{6}n\,(n+1)(2n+1)=\frac{1}{6}(2n^3+3n^2+n),$$

$$S_3(n)=\left\{\frac{n(n+1)}{2}\right\}^2=\frac{1}{4}(n^4+2n^3+n^2)$$

were also familiar to the Wasanists, and the general expression for $S_k(n)$ was studied with care by them, who showed their unusual power in this line also. The method taken by them was to find a recurring formula by which $S_{r+1}(n)$ is deduced from $S_r(n)$. To make the description simpler, let us put

$$(6) \qquad S_k(n) \equiv H_k(n)/d_k,$$

$H_k(n)$ being a polynomial with integral coefficients, and d_k an integer independent of n.

The Wasanists used first to find $H_k(n)$, and then to determine d_k as the algebraic sum of the coefficients in $H_k(n)$. This way of determination of d_k is verified by putting $n=1$ in (6), since $S_k(1)=1$ and $H_k(1)$ is the algebraic sum of the coefficients in $H_k(n)$. So we will proceed to describe how the Wasanists were successful in finding successively the coefficients in $H_2(n)$, $H_3(n)$,······. We believe that the reader of this paper will not fail to recall in his mind the labour of James Bernoulli in this direction.

First let us tabulate the coefficients in $H_1(n)$, $H_2(n)$,····· as follows.

	1	0						
$H_1(n)$	1	1	0					
$H_2(n)$	2	3	1	0				
$H_3(n)$	1	2	1	0	0			
$H_4(n)$	6	15	10	0	-1	0		
$H_5(n)$	2	6	5	0	-1	0	0	
$H_6(n)$	6	21	21	0	-7	0	1	0
......	....							

As often done before, let us again denote these numbers by $f_{i,k}$. For the sake of illustration, we will begin with some examples.

To find the coefficients in $H_5(n)$ from those in $H_4(n)$.

$$f_{61}'=f_{51}\times 6\div 6=\quad 6\times 6\div 6=\quad 6,$$
$$f_{62}'=f_{52}\times 6\div 5=\ 15\times 6\div 5=\ 18,$$
$$f_{63}'=f_{53}\times 6\div 4=\ 10\times 6\div 4=\ 15,$$
$$f_{64}'=f_{54}\times 6\div 3=\quad 6\times 6\div 3=\quad 0,$$
$$f_{65}'=f_{55}\times 6\div 2=-1\times 6\div 2=-3,$$
$$f_{66}'=f_{56}\times 6\div 1=\quad 0\times 6\div 1=\quad 0,$$
$$f_{67}'=0.$$

Removing the common factor 2([1]), we get

$$f_{61}=2,\qquad f_{62}=6,\qquad f_{63}=5,\qquad f_{64}=0,$$
$$f_{65}=-1,\quad f_{66}=0,\qquad f_{67}=0,$$

whence follows that

$$H_5(n)=2n^6+6n^5+5n^4-n^3,$$
$$d_5=2+6+5-1=12.$$

Hence we get

$$S_5(n)\equiv\frac{1}{12}(2n^6+6n^5+5n^4-n^3).$$

To find the coefficients in $H_6(n)$ from those in $H_5(n)$.

$$6+1=7,$$
$$f_{71}'=f_{61}\times 7\div 7=\quad 2\times 7\div 7=\quad 2,$$
$$f_{72}'=f_{62}\times 7\div 6=\quad 6\times 7\div 6=\quad 7,$$
$$f_{73}'=f_{63}\times 7\div 5=\quad 5\times 7\div 5=\quad 7,$$
$$f_{74}'=f_{64}\times 7\div 4=\quad 0\times 7\div 4=\quad 0,$$
$$f_{75}'=f_{65}\times 7\div 3=-1\times 7\div 3=-\frac{2}{3},$$
$$f_{76}'=f_{66}\times 7\div 2=0\times 7\div 2=0,$$
$$f_{77}'\quad\text{not determined here,}$$
$$f_{78}'=0.$$

Clearing the fractions, we get

$$f_{71}=6,\qquad f_{72}=21,\qquad f_{73}=21,\qquad f_{74}=0,$$
$$f_{75}=-7,\quad f_{76}=0,\qquad\quad *\quad ,\qquad f_{78}=0,$$

and then

$$f_{77}=f_{72}-(f_{71}+f_{73}+f_{74}+f_{75}+f_{76}+f_{78})=1.$$

([1]) Evidently this is not necessary.

Hence

$$H_6(n) \equiv 6n^7 + 21n^6 + 21n^5 - 7n^3 + n,$$
$$d_6 = 6 + 21 + 21 - 7 + 1 = 52,$$
$$S_6(n) = \frac{1}{52}(6n^4 + 21n^4 + 21n^3 - 7n^2 + n).$$

Now we will pass on to the general case to find $H_r(n)$. By putting

$$f'_{r+1,1} = f_{r,1} \times (r+1) \div (r+1) = f_{r,1},$$
$$f'_{r+1,2} = f_{r,3} \times (r+1) \div r,$$
$$f'_{r+1,3} = f_{r,3} \times (r+1) \div (r-1),$$
$$f'_{r+1,4} = f_{r,4} \times (r+1) \div (r-2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f'_{r+1,\,r+1} = f_{r,\,r+1} \times (r+1) \div 1,$$
$$f'_{r+1,\,r+2} = 0,$$

form the sequence

$$f'_{r+1,1}, \quad f'_{r+1,2}, \quad f'_{r+1,3}, \cdots\cdots, f'_{r+1,\,r+2},$$

and clear the fractions, remove the common factors, and denote the results by

$$f_{r+1,1}, \quad f_{r+1,2}, \quad f_{r+1,3}, \cdots\cdots, f_{r+1,\,r+2},$$

for the even suffix $r+1$. If $r+1$ be odd, the coefficient $f_{r+1,\,r+1}$ of the first power of n, is given by

$$(7) \quad f_{r+1,\,r+1} = f_{r+1,2} - (f_{r+1,1} + f_{r+1,3} + f_{r+1,4} + \cdots\cdots + f_{r+1,\,r} + f_{r+1,\,r+2}).$$

In both cases we have

$$d_{r+1} = 2f_{r+1,2}$$

This is the method described in Sanpô Zenkyô. Though in the above calculation we have interwoven the values of

$$f_{r+1,4}, \quad f_{r+1,6}, \quad f_{r+1,8}, \quad \cdots\cdots,$$

which are all zero, in order to show their relation to the others, the Wasanists did not do so, but put them equal to zero from the beginning. The relation (7) is obvious if we observe that

$$2f_{r+1,2} = f_{r+1,1} + f_{r+1,2} + f_{r+1,3} + \cdots\cdots + f_{r+1,\,r+1} + f_{r+1,\,r+2}.$$

Anmei Aida gave in his Sanpô Dajutu the following rule not essentially different from the foregoing.

(i) From $nH_1(n)$ we get $n^3 + n^2$. Divide its terms by 3, 2 respectively and get

$$\frac{n^3}{3}+\frac{n^2}{2}.$$

Clearing the fractions, we get

$$2n^3+3n^2,$$

and add A_3n, where A_3 is the coefficient of the second term diminished by the sum of all other coefficients, namely

$$A_3=3-2=1.$$

Thus we get at last

$$H_3(n)\equiv 2n^3+3n^2+n, \qquad d_3=2+3+1=6.$$

(ii) From $nH_2(n)$ we get $2n^4+3n^3+n^2$. Divide its terms by 4, 3, 2 respectively and get

$$\frac{2n^4}{4}+\frac{3n^3}{3}+\frac{n^2}{2}.$$

Clearing the fractions, we get

$$n^4+2n^3+n^2,$$

and add A_4n, where A_4 is the coefficient of the second term diminished by the sum of all other coefficients, namely

$$A_4=2-(1+1)=0.$$

Thus we get at last

$$H_3(n)\equiv n^4+2n^3+n^2, \qquad d_3=1+2+1=4.$$

(iii) From $nH_3(n)$ we get $n^5+2n^4+n^3$. Divide its terms by 5, 4, 3, 2 respectively and get

$$\frac{n^5}{5}+\frac{2n^4}{4}+\frac{n^3}{3},$$

Clearing the fractions, we get

$$6n^5+15n^4+10n^3,$$

and add A_5n, where A_5 is equal to the coefficient of the second term diminished by the sum of all other coefficients, namely

$$A_5=15-(6+10)=-1.$$

Thus we get at last

$$H_4\equiv 6n^5+15n^4+10n^3-n, \qquad d_4=6+15+10-1=36.$$

Etc., etc..

The validity of these two methods can be shown by the formula given by James Bernoulli in his *Ars Conjectandi* posthumously pub-

lished at Basel in 1713, though the proofs of the Wasanists are unknown to us.

By Bernoulli's formula we get

$$(8)\quad \frac{S_{2k}(n)}{(2k)!}=\frac{n^{2k+1}}{(2k+1)!}+\frac{n^{2k}}{2(2k)!}+\frac{B_1}{2\,!}\,\frac{n^{2k-1}}{(2k-1)!}-\frac{B_4}{4\,!}\,\frac{n^{2k-3}}{(2k-3)!}+\cdots+(-1)^{k-1}\,\frac{B_k}{(2k)!}\,\frac{n}{2\,!},$$

$$(9)\quad \frac{S_{2k+1}(n)}{(2k+1)!}=\frac{n^{2k+2}}{(2k+2)!}+\frac{n^{2k+1}}{2(2k+1)!}+\frac{B_1}{2\,!}\,\frac{n^{2k}}{(2k)\,!}-\frac{B_2}{4\,!}\,\frac{n^{2k-2}}{(2k-2)!}+\cdots\cdots+(-1)^{k-1}\frac{B_k}{(2k)!}\,\frac{n^2}{2\,!},$$

$$(10)\quad \frac{S_{2k+2}(n)}{(2k+2)!}=\frac{n^{2k+3}}{(2k+3)!}+\frac{n^{2k+2}}{2(2k+1)!}+\frac{B_1}{2\,!}\,\frac{n^{2k+1}}{(2k+1)!}-\frac{B_2}{4\,!}\,\frac{n^{2k-1}}{(2k-1)!}+\cdots\cdots+(-1)^{k-1}\,\frac{B_k}{(2k)!}\,\frac{n^3}{4\,!}+(-1)^k\frac{B_{k+1}}{(2k+2)!}\,\frac{n}{2\,!}.$$

If we multiply each term of the second member of (8) by n, and divide the term of n^t by $t+1$, we get the second number of (9), and if we diminish the coefficient of the second term of the second member of (8) by the sum of all the other coefficients, we have

$$\frac{1}{2(2k)!}-\left[\frac{1}{(2k+1)!}+\frac{B_1}{2!(2k-1)!}-\frac{B_2}{4!(2k-3)!}+\cdots\cdots+(-1)^{k-1}\frac{B_n}{(2k)!2!}\right]$$
$$=\frac{1}{2(2k)!}-\left[\frac{S_{2k}(1)}{(2k)!}-\frac{1}{2(2k)!}\right]$$
$$=0,\qquad (\because\ S_i(1)=1).$$

If we multiply each term of the second member of (9) by n, and divide the term of n^s by $s+1$, we get the second member of (10), the last term being excluded. If we diminish the coefficient of the second term of the second member of (9) by the sum of all the other coefficients, we have

$$\frac{1}{2(2k+2)!}-\left[\frac{1}{(2k+3)!}+\frac{B_1}{2!(2k+1)!}-\frac{B_2}{4!(2k-1)!}+\cdots\cdots+(-1)^{k-1}\frac{B_k}{(2k)!\,4!}\right]$$
$$=\frac{1}{2(2k+2)!}-\left[\frac{S_{2k+2}(1)}{(2k+2)!}-\frac{1}{2(2k+2)!}-(-1)^k\frac{B_{k+1}}{(2k+2)!2!}\right.$$
$$=(-1)^k\frac{B_{k+1}}{(2k+2)!2!}.$$

Hence the first member of this equality is certainly the coefficient of the first power of n in the second member of (10). Hence the foregoing methods of the Wasanists are verified.

Other methods of finding $H_r(n)$ known to the Wasanists.

1°. As an application of binomial coefficients.

In Kowa Seki's 關孝和 Kwatuyô Sanpô 括要算法, published posthumously in 1709, is found the following.

From $(n+1)^2-1$, $(n+1)^3-1$, $(n+1)^4-1, \cdots\cdots$ we construct the following table.

$$\binom{2}{0},\ \binom{2}{1},\ 0,$$

$$\binom{3}{0},\ \binom{3}{1},\ \binom{3}{2},\ 0,$$

$$\binom{4}{0},\ \binom{4}{1},\ \binom{4}{2},\ \binom{4}{3},\ 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

(11) $$\binom{2k+1}{0}, \binom{2k+1}{1}, \binom{2k+1}{2}, \binom{2k+1}{3}, \binom{2k+1}{4}, \cdots\cdots \binom{2k+1}{2k-1}, \binom{2k+1}{2k}, 0,$$

(12) $$\binom{2k+2}{0}, \binom{2k+2}{1}, \binom{2k+2}{2}, \binom{2k+2}{3}, \binom{2k+2}{4}, \cdots\cdots \binom{2k+2}{2k-1}, \binom{2k+2}{2k}, \binom{2k+2}{2k+1}, 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Multiply	the	1–st	column	by	1,
,,	,,	2–nd	,,	,,	1/2,
,,	,,	3–rd	,,	,,	1/6,
,,	,,	5–th	,,	,,	−1/30,
,,	,,	7–th	,,	,,	1/42,
,,	,,	9–th	,,	,,	−1/30,
,,	,,	11–th	,,	,,	5/66,
....					

and multiply all the 4–th, 6–th, 8–th,$\cdots\cdots$by zero.

After reducing them to the fractions having common denominator D_r, we get the coefficients in $H_r(n)$, and then d_r is obtained by

(13) $$d_r = (r+1)D_r.$$

The successive multipliers for respective columns above mentioned were derived by the Syôsahô 招差法, a method in Wasan resembling to that of finite difference and theory of arithmetic series of higher orders, and

in absolute values, they are nothing but Bernoulli's numbers. Since we have for $S_r(n)$ by Bernoulli's formula

$$(r+1)S_r(n)=n^{r+1}+\frac{1}{2}\binom{r+1}{1}n^r+\binom{r+1}{2}B_1n^{r-1}-\binom{r+1}{4}B_2n^{r-3}$$

$$+\binom{r+1}{6}B_3n^{r-5}\cdots\cdots\cdots\cdots\cdots\cdots,$$

the relation (13) is obvious. Hence we see that this formula of Bernoulli and the method described in Kwatuyô Sanpô are identical. The resembling circumstances under which they were published, and the very proximity of their dates of issue, look very strange to us.

Anmei Aida stated in his Sanpô Dajutu the foregoing method in the following form: Arrange the coefficients of n in

$$(n+1)^2-1,\quad (n+1)^3-1,\quad (n+1)^4-1,\quad \cdots\cdots\cdots$$

as shown on page 317.

Multiply all the numbers in the 2nd, 3rd, 4th, 5th, 6th$\cdots\cdots$ columns by $\frac{1}{2}$, $\frac{1}{6}$, 0, $-\frac{1}{30}$, 0, $\frac{1}{42}$, $\cdots\cdots$ respectively, which were found as follows:

$$r=2:\quad \left[r-\binom{2}{0}\right]\div\binom{2}{1}=\frac{1}{2},$$

$$r=3:\quad \left[r-\left\{\binom{3}{0}+\frac{1}{2}\binom{3}{1}\right\}\right]\div\binom{3}{2}=\frac{1}{6},$$

$$r=4:\quad \left[r-\left\{\binom{4}{0}+\frac{1}{2}\binom{4}{1}+\frac{1}{6}\binom{4}{2}\right\}\right]\div\binom{4}{3}=0,$$

$$r=5:\quad \left[r-\left\{\binom{5}{0}+\frac{1}{2}\binom{5}{1}+\frac{1}{6}\binom{5}{2}+0.\binom{5}{3}\right\}\right]\div\binom{5}{4}=-\frac{1}{30},$$

$$r=6:\quad \left[r-\left\{\binom{6}{0}+\frac{1}{2}\binom{6}{1}+\frac{1}{6}\binom{6}{2}+0.\binom{6}{3}-\frac{1}{30}\binom{6}{4}\right\}\right]\div\binom{6}{5}=0,$$

$$r=7:\quad \left[r-\left\{\binom{7}{0}+\frac{1}{2}\binom{7}{1}+\frac{1}{6}\binom{7}{2}+0.\binom{7}{3}-\frac{1}{30}\binom{7}{4}+0.\binom{7}{5}\right\}\right]=\frac{1}{42}.$$

Then tabulate these results, and clear the fractions with respect to each row:

1	1	0				for	$H_1(n)$,
2	3	1	0			,,	$H_2(n)$,
1	2	1	0	0		,,	$H_3(n)$,
6	15	10	0	-1	0	,,	$H_4(n)$,

2	6	5	0	-1	0	0		,,	$H_5(n)$,
6	21	21	0	-7	0	1	0	,,	$H_6(n)$,
.....									.....

Hence, for example, we have

$$H_5(n)=2n^6+6n^5+5n^4-n^2,$$

and for d_5 Aida took twice the coefficient of the second term, namely 6.

This process can be verified as follows: The comparison of

$$(14)\qquad (2k+1)S_{2k}(n)=n^{2k+1}+\frac{1}{2}\binom{2k+1}{1}n^{2k}+\binom{2k+1}{2}B_1n^{2k-1}$$

$$-\binom{2k+1}{4}B_1n^{2k-3}+\binom{2k+1}{6}B_3n^{2k-5}+\cdots$$

$$\cdots+(-1)^{k-1}\binom{2k+1}{2k}B_kn,$$

$$(15)\qquad (2k+2)S_{2k+1}(n)=n^{3k+2}+\frac{1}{2}\binom{2k+2}{1}n^{3k+1}+\binom{2k+2}{2}B_1n^{2k}$$

$$-\binom{2k+2}{4}B_3n^{2k-1}+\binom{2k+2}{6}B_3n^{2k-4}-\cdots$$

$$\cdots+(-1)^{k-1}\binom{2k+2}{2k}B_kn^2,$$

shows at once that $(-1)^{k-1}B_k$ is the multiplier corresponding to $(2k+1)$-th column. But since $S_r(1)=1$, we get from (14)

$$2k+1=1+\frac{1}{2}\binom{2k+1}{1}+\binom{2k+1}{2}B_1-\binom{2k+1}{4}B_2+\binom{2k+1}{6}B_3$$

$$\cdots\cdots+(-1)^{k-2}\binom{2k+1}{2k-2}B_{k-1}+(-1)^{k-1}\binom{2k+1}{2k}B_k,$$

whence follows that

$$(-1)^{k-1}B_k=\left[(2k+1)-\left\{1+\frac{1}{2}\binom{2k+1}{1}+\binom{2k+1}{2}B_1+\cdots\right.\right.$$

$$\left.\left.\cdots+(-1)^{k-2}\binom{2k+1}{2k-2}B^{k-1}\right\}\right]\div\binom{2k+1}{2k}.$$

From (12), (13) we see that the multiplier corresponding to the $(2k+2)$-th column is zero. But from (15) we have

$$2k+2=1+\frac{1}{2}\binom{2k+2}{1}+\binom{2k+2}{2}B_1-\binom{2k+2}{4}B_2+\cdots\cdots+(-1)^{k-1}\binom{2k+2}{2k}B_k.$$

Hence we have

$$0=(2k+2)-\left\{1+\frac{1}{2}\binom{2k+2}{1}+\binom{2k+2}{2}B_1-\binom{2k+2}{4}B_2+\cdots\right.$$
$$\left.\cdots+(-1)^{k-1}\binom{2k+2}{4k}B_k\right\}.$$

Thus Aida's method is verified.

Now if we denote the coefficients in $H_n(n)$ on page 312 by $h_{i,k}$, and put $r+2=l$, the relation recorded in Matunaga's Sanpô Zenkyô can be written in the form

$$h_{r,1}=1,$$
$$h_{r,2}=2^{r+1}-l,$$
$$h_{r,3}=3^{r+1}-\left\{2^{r+1}-\frac{1}{2}(l-1)\right\}l,$$
$$h_{r,4}=4^{r+1}-\left[3^{r+1}-\frac{1}{2}\left\{2^{r+1}-\frac{1}{3}(l-2)\right\}(l-1)\right]l,$$
$$h_{r,5}=5^{r+1}-\left(4^{r+1}-\frac{1}{2}\left[3^{r+1}-\frac{1}{3}\left\{2^{r+1}-\frac{1}{4}(l-3)\right\}(l-2)\right](l-1)\right)l,$$

....

2°. To deduce $H_r(n)$ from a set of binomial expansions.

If $r=2s+1$, take

$$(n+1)^{r+2},\quad (n+1)^{r+1},\quad (n+1)^r,\quad (n+1)^{r-2},\quad (n+1)^{r-4},\ \cdots\cdots\ (n+1),$$

and if $r=2s+2$, take

$$(n+1)^{r+2},\quad (n+1)^{r+1},\quad (n+1)^r,\quad (n+1)^{r-2},\quad (n+1)^{r-4},\ \cdots\cdots\ (n+1)^2,$$

and denote the expansion of $(n+1)^p$ in descending powers of n by $(n+1)_0^p$. Then eliminate the second term of $(n+1)_0^{r+1}$ between $(n+1)_0^{r+2}$, $(n+1)_0^{r+1}$, and get the eliminant $E_1(n)$ with integral coefficients. Eliminate between $E_1(n)$, $(n+1)_0^{r-1}$ the second term of $(n+1)_0^{r-1}$, and get the eliminant $E_2(n)$ with integral coefficients. If we continue this process, we will finally arrive at $H_{r+1}(n)$. Since $d_{r+1}=H_{r+1}(1)$, we have thus

$$S_{r+1}(n)=H_{r+1}(n)/H_{r+1}(1).$$

Examples. To find $S_3(n)$, $3+1=4$,

$$(n+1)^4=n^4+4n^3+6n^2+4n+1,$$
$$(n+1)^3=\qquad n^3+3n^2+3n+1,$$

whence

$$E_1(n)=(n+1)^4-2(n+1)^3$$
$$=n^4+2n^3+0-2n-1,$$

which combined with $(n+1)^2=n^2+2n+1$, gives by elimination of n

$$E_2(n)=n^4+2n^3+n^2, \qquad d_3=1+2+1=4,$$

so that

$$S_3(n)=\frac{1}{4}(n^4+2n^3+n^2).$$

To find $S_4(n)$,

$$\begin{aligned}(n+1)^5&=n^5+5n^4+10n^3+10n^2+5n+1,\\ (n+1)^4&=\qquad n^4+\ 4n^3+\ 6n^2+4n+1,\end{aligned}$$

whence

$$\begin{aligned}E_1(n)&=2(n+1)^5-5(n+1)^4\\ &=2n^5+5n^4+0-10n^2-10n-3,\\ (n+1)^3&=\qquad\qquad n^3+\ 3n^2+\ 3n+1\,;\\ E_2(n)&=6n^5+15n^4+10n^3+0+0+1,\\ n+1&=\qquad\qquad\qquad n+1\,;\\ E_3(n)&=6n^5+15n^4+10n^3-n,\\ d_3&=\ 6\ +15\ \ +10\ \ -1=30.\end{aligned}$$

Hence we get

$$S_4(n)=\frac{1}{30}(6n^5+15n^4+10n^3-n).$$

The foregoing method is recorded in Daseki Syûhô and Sanpô Zenkyô. In the former it is remarked that the signs of the coefficients of $(n+1)^{s+1}$, $(n+1)^s$,$\cdots\cdots$ are alternately different, if we keep the coefficient of the highest term of each $E(n)$ always positive. The tabulation of these results gives

$$\begin{aligned}&(n+1)^2-\ (n+1),\\ &2(n+1)^3-\ 3(n+1)^2+\ (n+1),\\ &(n+1)^4-\ 2(n+1)^3+\ (n+1)^2,\\ &6(n+1)^5-15(n+1)^4+10(n+1)^3-\ (n+1),\\ &2(n+1)^6-\ 6(n+1)^5+\ 5(n+1)^4-\ (n+1)^2,\\ &6(n+1)^7-21(n+1)^6+21(n+1)^5-\ 7(n+1)^3+(n+1),\\ &3(n+1)^8-12(n+1)^7+14(n+1)^6-\ 7(n+1)^4+\ (n+1)^2,\\ &10(n+1)^9-45(n+1)^8+60(n+1)^7-42(n+1)^5+20(n+1)^3-3(n+1),\\ &2(n+1)^{10}-10(n+1)^9+15(n+1)^8-14(n+1)^6+10(n+1)^4-3(n+1)^2,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

If we change the sign minus in the second column into plus, the arrangement of the coefficients is just the same as that on page 312.

From Bernoulli's formula

$$(r+1)S_r(n)=n^{r+1}+\frac{1}{2}\binom{r+1}{1}n^r+\binom{r+1}{2}B_1n^{r-1}-\binom{r+1}{4}B_2n^{r-3}+\cdots\cdots,$$

we have

$$(r+1)S_r(n+1)=(n+1)^{r+1}+\frac{1}{2}\binom{r+1}{1}(n+1)^r+\binom{r+1}{2}B_1(n+1)^{r-1}$$
$$-\binom{r+1}{1}B_2(n+1)^{r-3}+\cdots\cdots,$$

while on the other hand we have

$$(r+1)S_r(n)+(r+1)(n+1)^r=(r+1)S_r(n+1).$$

Hence we get

$$(r+1)S_r(n)=-(r+1)(n+1)^r+(n+1)^{r+1}+\frac{1}{2}\binom{r+1}{1}(n+1)^r$$
$$+\binom{r+1}{2}B_1(n+1)^{r-1}-\cdots\cdots$$
$$=(n+1)^{r+1}-\frac{1}{2}\binom{r+1}{1}(n+1)^r+\binom{r+1}{2}B_1(n+1)^{r-1}\cdots\cdots.$$

This shows the meaning of the arrangement of the coefficients in the above table.

3. To express $H_r(n)$ in terms of figurate numbers.

The Wasanists were also successful in deriving the general Hôda formula from figurate numbers.

For example

$$1.\,a_{2,n}+1.\,a_{2,n-1}=n(n+1)(n+2)/3!+(n-1)n(n+1)/3!$$
$$=n(n+1)(2n+1)/3!=S_2(n),$$
$$1.\,a_{3,n}+4a_{3,n-1}+1.\,a_{3,n-2}=n(n+1)(n+2)(n+3)/4!$$
$$+4(n-1)(n+1)(n+2)/4!+(n-2)(n-1)n(n+1)/4!$$
$$=\{n(n+1)/2\}^2=S_3(n).$$

....

Thus the theorem which E. Locchi showed as a new expression in his "Über die Summe der Potenzen der Natürlichen Zahlen"(1) was known to the Wasanists as early as in the first half of the eighteenth century.

(1) Monatshefte für Mathematik und Physik, vol. 4, 1893.

To find $H_r(n)$, being considered as expressed linearly in terms of $a_{r,s}$, we must determine the coefficients of $a_{r,n}$, $a_{r,n-1}$, $a_{r,n-2}$, $\cdots\cdots$; thus

1						for	$H_1(n)$,
1	1					,,	$H_2(n)$,
1	4	1				,,	$H_3(n)$,
1	11	11	1			,,	$H_4(n)$,
1	26	66	26	1		,,	$H_5(n)$,
1	57	302	302	57	1	,,	$H_6(n)$,
......						......	

If we denote the numbers in the k-th column of this table by A_{1k}, A_{2k}, A_{2k}, $\cdots\cdots$, we find in Daseki Syûhô the relation

$$A_{i+1,\,k+1}=(i+1)A_{i+1,\,k}+(k+1)A_{i,\,k+1}, \tag{16}$$

by which we can calculate A_{ik} successively, so that we may express $H_r(n)$ in terms of figurate numbers.

Next we will reproduce Locchi's proof with a slight modification in symbols.

If we put $k^r=u_k$, by the well known formula, we will have

$$S_r(n)=\binom{n}{1}+\Delta u_1\binom{n}{1}+\Delta^2 u_1\binom{n}{3}+\Delta^3 u_1\binom{n}{4}+\cdots\cdots+\Delta^r u_1\binom{n}{r+1}.$$

For $r=1$, we get $\Delta u_1=1$, $\Delta^2 u_1=0$, so that

$$S_1(n)=\binom{n}{1}+\binom{n}{2}=\binom{n+1}{2}=a_{2,\,n}.$$

For $r=2$, we get $\Delta u_1=3$, $\Delta^2 u_1=2$, $\Delta^3 u_1=0$, so that

$$S_2(n)=\binom{n}{1}+3\binom{n}{2}+2\binom{n}{3}=\binom{n+2}{3}+\binom{n+1}{3}$$
$$=a_{3,\,n+2}+a_{3,\,n+1}.$$

For $r=3$, we get $\Delta u_1=7$, $\Delta^2 u_1=1$, $\Delta^3 u_1=6$, $\Delta^4 u_1=0$, so that

$$S_3(n)=\binom{n}{1}+7\binom{n}{2}+12\binom{n}{3}+6\binom{n}{4}$$
$$=\binom{n+3}{4}+4\binom{n+2}{4}+\binom{n+1}{4}$$
$$=a_{4,\,n+3}+4a_{4,\,n+2}+a_{4,\,n+1}.$$

Continuing in this way, we have in general

$$S_r(n)=A_{r,1}\binom{n+r}{r+1}+A_{r-1,\,2}\binom{n+r-1}{r+1}+A_{r-2,\,3}\binom{n+r-1}{r+1}+\cdots\cdots+A_{1,r}\binom{n+1}{r+1}$$

where $A_{r,1}=1$ and the $A_{m,\,n}$'s are constants determined as below.

If we remember that

$$\Delta u_1 = 2^r - 1,$$
$$\Delta^2 u_1 = 3^r - 2\cdot 2^r + 1,$$
$$\Delta^3 u_1 = 4^r - 3\cdot 3^r + 3\cdot 2^r - 1,$$
$$\Delta^4 u_1 = 5^r - 4\cdot 4^r + 6\cdot 3^r - 4\cdot 2^r + 1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

we get

$$A_{r-1,\,2} = \Delta u_1 - r = 2^r - 1 - r = 2^r - \binom{r+1}{1}$$

$$\begin{aligned} A_{r-2,\,3} &= \Delta^2 u_1 - (\Delta u_1 - 1) - (\Delta u_1 - 2) - (\Delta u_1 - 3) - \cdots\cdots - (\Delta u_1 - \overline{r+1}) \\ &= \Delta^2 u_1 - (r-1)\Delta u_1 - (1+2+3+\cdots\cdots+\overline{r-1}) \\ &= 3^r - 2\cdot 2^r + 1 - (r-1)(2^r-1) + \binom{r}{2} \\ &= 3^r - (r+1)2^r + \binom{r}{1} + \binom{r}{2} \\ &= 3^r - \binom{r+1}{1}2^r + \binom{r+1}{2}, \end{aligned}$$

and in general

$$A_{r-i+1,\,i} = i^r - \binom{r+1}{1}(i-1)^r + \binom{r+1}{2}(i-2)^r - \cdots\cdots + (-1)^{i-1}\binom{r+1}{i+1},$$

hence we have

$$A_{r-i,\,i+1} = (i+1)^r - \binom{r+1}{1}i^r + \binom{r+1}{2}(i-1)^r - \cdots\cdots + (-1)^i\binom{r+1}{i},$$

$$A_{r-i+1,\,i+1} = (i+1)^{r+1} - \binom{r+2}{1}i^{r+1} + \binom{r+2}{2}(i-1)^{r+1} - \cdots\cdots + (-1)^i\binom{r+2}{1}.$$

From the last two formulae we get

$$\begin{aligned} &(r-i+1)A_{r-i+1,\,i} + (i+1)A_{r-i,\,i+1} \\ &\qquad = (i+1)^{r+1} + \sum_{s=0}^{i-1}(-1)^s(i-s)^r\left[(r-i+1)\binom{r+1}{s} + (i+1)\binom{r+1}{s+1}\right] \\ &\qquad = (i+1)^{r+1} + \sum_{s=0}^{i-1}(-1)^{s+1}(i-s)^{r+1}\binom{r+2}{s+1} \\ &\qquad = A_{r-i+1,\,i+1}, \end{aligned}$$

which is nothing but the relation (16).

On the Roots of the Algebraic Equation of the Form
$$f+k_1f'+k_2f''+\cdots\cdots+k_nf^{(n)}=0,$$

by

Yoshimichi Uchida, Sendai.

The following theorem is usually due to Hermite or De Longchamps(1): If all the roots of

$$f\equiv x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots\cdots+a_{n-1}x+a_n=0 \qquad (1)$$

are real, then a sufficient condition that the roots of

$$f+k_1f'+k_2f''+\cdots\cdots+k_nf^{(n)}=0$$

are all real, is that all the roots of the equation

$$x^n+k_1x^{n-1}+k_2x^{n-2}+\cdots\cdots+k_n=0$$

are real, $a_1, a_2, \cdots\cdots, a_n$ and $k_1, k_2, \cdots\cdots, k_n$ being all real.

By putting $f=x^n$ we can deduce the theorem: If equation (1) has no imaginary roots, then the equation

$$F\equiv x^n+na_1x^{n-1}+n(n-1)a_2x^{n-2}+\cdots\cdots+n!\,a_n=0$$

has no imaginary roots. This theorem stands in an intimate relation to that proposed by Laguerre(2) to solve.

From these theorems we can prove the theorem: The necessary and sufficient condition that the equation

$$F+k_1F'+k_2F''+\cdots\cdots+k_nF^{(n)}=0 \qquad (2)$$

has no imaginary roots, is that the equation

$$x^n+nk_1x^{n-1}+n(n-1)k_2x^{n-2}+\cdots\cdots+n!\,k_n=0 \qquad (3)$$

has no imaginary roots.

To prove this theorem: By developing and rearranging the terms of equation (2) we get the equation of the form

$$\varphi+a_1\varphi'+a_2\varphi''+\cdots\cdots+a_n\varphi^{(n)}=0,$$

where

---

(1) Hermite, Nouvelles Annales de Mathématiques, série 2, t. 5, 1866, p. 479; De Lonchamps, Journal de Mathématiques spéciales, Question No. 278 (see Laisant, Recueil de problèmes—Algèbre. 1895, p. 82).

(2) Laguerre, Nouvelles Annales de Mathématiques, série 3, t. 1, 1882, p. 142.

$$\varphi \equiv x^n + nk_1 x^{n-1} + \cdots\cdots + n!\, k_n.$$

Hence by Hermite's theorem, when all the roots of equation (3) are real, the sufficiency of the condition follows. The necessity of the condition is easily shown, since by putting $F \equiv x^n$ equation (2) is transformed into

$$x^n + nk_1 x^{n-1} + n(n-1)k_2 x^{n-2} + \cdots\cdots + n!\, k_n = 0,$$

which is (3) itself.

By a similar way we can prove the theorem: The necessary and sufficient condition that the equation

$$F + k_1 F' + k_2 F'' + \cdots\cdots + k_r F^{(r)} = 0 \qquad (r \leqq n) \tag{4}$$

has no imaginary roots is that the equation

$$x^r + nk_1 x^{r-1} + n(n-1)k_2 x^{k-2} + \cdots\cdots + n(n-1)\cdots(n-r+1)k_r = 0 \tag{5}$$

has no imaginary roots.

To prove this theorem; equation (4) may be written, by putting

$$\psi \equiv x^n + nk_1 x^{n-1} + \cdots\cdots + n(n-1)\cdots(n-r+1)k_r x^{n-r},$$

in the form

$$\psi + a_1 \psi' + a_2 \psi'' + \cdots\cdots + a_n \psi^{(n)} = 0.$$

Hence if the equation $\psi = 0$ has no imaginary roots, the sufficiency of the condition follows from Hermite's theorem. Next by putting $F \equiv x^n$, equation (4) is transformed into

$$x^{n-r}\{x^r + nk_1 x^{r-1} + n(n-1)k_2 x^{r-2} + \cdots\cdots + n(n-1)\cdots(n-r+1)k_r\} = 0,$$

which is (5), merely multiplied by a power of x. Hence the condition is necessary.

From the above two theorems we have the theorem: If

$$f \equiv x^n + a_1 x^{n-1} + \cdots\cdots + a_n = 0$$

has no imaginary roots, and if

$$F \equiv x^n + na_1 x^{n-1} + \cdots\cdots + n!\, a_n,$$

the necessary and sufficient condition that the equation

$$F + k_1 F' + k_2 F'' + \cdots\cdots + k_r F^{(r)} = 0 \qquad (r \leqq n)$$

has no imaginary roots is that the equation

$$x^r + nk_1 x^{r-1} + n(n-1)k_2 x^{r-2} + \cdots\cdots + n(n-1)\cdots(n-r+1)k_r = 0$$

has no imaginary roots, $a_1, a_2, \cdots\cdots, a_n$ and $k_1, k_2, \cdots\cdots, k_n$ being all real.

We can also prove that the integration constants in the equation

$$\overbrace{\int\int\int\cdots\int}^{m} F dx = 0 \tag{6}$$

can be so chosen that the equation has real roots only.

For,

$$\overbrace{\int\int\int\cdots\int}^{m} F dx = \int\int\int\cdots\int \{x^n + na_1x^{n-1} + n(n-1)a_2x^{n-2} + \cdots\cdots + n!\, a_n\} dx$$

$$= \frac{x^{m+n}}{(m+n)\cdots(n+1)} + \frac{a_1x^{m+n-1}}{(m+n-1)\cdots(n+1)} + \cdots\cdots$$

$$+ \frac{n!\, a_n x^m}{m!} + c_1x^{m-1} + \cdots\cdots + c_m$$

$$= \frac{n!}{(m+n)!}\{x^{m+n} + (m+n)a_1x^{m+n-1} + (m+n)(m+n-1)a_2x^{m+n-2}$$

$$+ \cdots\cdots + (m+n)\cdots(m+1)a_nx^m + c_1'x^{m-1} + \cdots + c_m'\}$$

$$= 0.$$

When $c_1' = c_2' = \cdots = c_m' = 0$, this equation becomes

$$n^{m+n} + (m+n)a_1x^{m+n-1} + (m+n)(m+n-1)a_2x^{m+n-2}$$

$$+ \cdots\cdots + (m+n)\cdots(m+1)a_nx^m = 0. \tag{7}$$

Now as equation (1), *i. e.*

$$f \equiv x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots\cdots + a_n = 0$$

has real roots only, the equation

$$x^{m+n} + a_1x^{m+n-1} + a_2x^{m+n-2} + \cdots\cdots + a_nx^m = 0$$

has also real roots only. Hence if we put

$$f = x^{m+n}$$

in Hermite's theorem, equation (7) has real roots only. Therefore we can choose integration constants so that equation (6) has real roots only.

On Some Algebraic Equations whose Roots are all Real and Distinct,

by

Yoshitomo Okada, Sendai.

Biehler proved the theorem: If the imaginary parts of all the roots of the equation of the nth degree

$$f(x) \equiv \varphi(x) + i\psi(x) = 0,$$

where $\psi(x)$ is of degree $n-1$, have the same sign, then all the roots of the equation $\varphi(x)=0$ of degree n are real and distinct, and are separated by the roots of the equation $\psi(x)=0$ which are also all real and distinct (Journal für reine und angewandte Mathematik 87, 1879, p. 350).

Auric proved the converse of this theorem (Comptes rendus de l'Académie des sciences, 137, 1903, p. 967) and its very simple proof has been given by Prof. Fujiwara (this Journal, Vol. 9, 1916, p. 105).

Applying the theorem and its converse, we will deduce some theorems on algebraic equations whose roots are all real and distinct.

1. Take the equations

$$f_j(x) \equiv \varphi_j(x) + i\psi_j(x) = 0, \qquad (j=1, 2, 3, \cdots\cdots, p),$$

each of which has such roots only that their imaginary parts have the same sign. Throughout this article we assume that the degree of φ_j is higher than that of ψ_j by 1, and the coefficients of the highest power of φ_j are of the same sign. If we choose p real numbers k_j such that the terms of the highest degrees of $k_j\psi_j(x)$ have the same sign, then the imaginary parts of the roots of all the equations

$$F_j(x) \equiv \varphi_j(x) + ik_j\psi_j(x) = 0$$

have the same sign. Hence, by Biehler's theorem, if we put

$$\prod_{j=1}^{p} \{F_j(x)\}^{m_j} \equiv \Phi(x) + i\Psi(x),$$

where m are positive integers, then all the roots of the equation $\Phi(x)=0$ are real and distinct, and are separated by the roots of the equation $\Psi(x)=0$ which are all real and distinct. Therefore we have

Theorem 1. If each of the equations

$$f_j(x) \equiv \varphi_j(x) + i\psi_j(x) = 0 \qquad (j=1, 2, 3, \cdots\cdots, p)$$

has such roots only that their imaginary parts have the same sign, and if for such real numbers k_j $(j=1, 2, 3, \cdots\cdots, p)$ (not zero) the terms of the highest degrees of the polynomials $k_j\psi_j(x)$ have the same sign, and for any positive integers m_j $(j=1, 2, 3, \cdots\cdots, p)$, the expression

$$\prod_{j=1}^{p} \{\varphi_j(x)+ik_j\psi_j(x)\}^{m_j} \equiv \Phi(x)+i\Psi(x),$$

be formed, then all the roots of the equation $\Phi(x)=0$ are real and distinct, and are separated by the roots of the equation $\Psi(x)=0$, which are all real and distinct.

By this theorem and the converse of Biehler's theorem, we have

Theorem 2. If $\varphi_j(x)=0$, $\psi_j(x)=0$ $(j=1, 2, 3 \cdots\cdots, p)$ be p sets of two equations with real coefficients, of degrees n and $n-1$ respectively, and if all the roots of the equation $\varphi_j(x)=0$ be real and distinct and be separated by the roots of the equation $\psi_j(x)=0$, which are all real and distinct, and if for such real numbers k_j $(j=1, 2, 3, \cdots\cdots, p)$ (not zero) the terms of the highest degrees of $k_j\psi_j(x)$ have the same sign and for any positive integers m_j $(j=1, 2, 3, \cdots\cdots, p)$, the expression

$$\prod_{j=1}^{p} \{\varphi_j(x)+ikj\,\psi_j(x)\}^{m_j} \equiv \Phi(x)+i\Psi(x),$$

be formed, then all of the roots of the equation $\Phi(x)=0$ are real and distinct, and are separated by the roots of the equation $\Psi(x)=0$ which are all real and distinct.

2. If we put specially $p=1$, $m_1=2$ in theorem 1, and if we drop the suffix 1, then, since $\Phi(x)=\varphi^2(x)-k^2\psi^2(x)$, the roots of the equation $\varphi^2(x)-k^2\psi^2(x)=0$ are all real and distinct. Hence the roots of the equation

$$\varphi(x)+k\psi(x)=0$$

are all real and distinct, k being any real number. Therefore we have the theorem: If the imaginary parts of all the roots of the equation of the nth degree

$$f(x)\equiv\varphi(x)+i\psi(x)=0$$

have the same sign, and if a and b be any two real numbers, then the roots of the equation

$$a\varphi(x)+b\psi(x)=0$$

are all real and distinct.

The above proof of this theorem is simpler than those given by Laguerre[1] and others.

---

[1] Laguerre, Journal für Mathematik 89, 1880, p. 339; Biehler, Hermite, etc.

Specializing theorem 2 in a similer manner we have the theorem: If $\varphi(x)=0$, $\psi(x)=0$ be two equations with real coefficients, of degrees n and $n-1$ respectively, and if all the roots of the equation $\varphi(x)=0$ are real and distinct, and are separated by the roots of the equation $\psi(x)=0$ which are all real and distinct, and if a and b be any two real numbers, then the roots of the equation

$$a\varphi(x)+b\psi(x)=0$$

are all real and distinct.

As a corollary of this theorem we have the result: If all the roots of the equation $\varphi(x)=0$ be real and distinct, then all the roots of the equation

$$a\varphi(x)+b\psi(x)=0$$

are real and distinct, where a and b are any real numbers.

From this result we can easily prove, as usual, the theorem which is due to Hermite or De Longchamps: If the roots of the equation of the nth degree $\varphi(x)=0$ are all real and distinct, and we take real numbers $k_1, k_2, \cdots\cdots, k_m$ $(m \leqq n)$ such that all the roots of the equation

$$x^m+k_1x^{m-1}+k_2x^{m-2}+\cdots\cdots+k_m=0$$

are real, then the roots of the equation

$$\varphi(x)+k_1\varphi'(x)+k_2\varphi''(x)+\cdots\cdots+k_m\varphi^{(m)}(x)=0$$

are all real and distinct.

If we put specially in theorem 2, $p=2$, $m_1=m_2=1$, $\psi_1(x)=\varphi_2(x)$ $=\varphi_1'(x)$ and $\psi_2(x)=\varphi_1''(x)$, and if we drop the suffix 1, then, for any real numbers k_1, k_2, (not zero) with the same sign,

$$\Phi(x)=\varphi(x)\varphi'(x)-k_1k_2\varphi'(x)\varphi''(x),$$
$$\Psi(x)=k_1\{\varphi'(x)\}^2+k_2\varphi(x)\varphi''(x);$$

whence we have the result: If k_1, k_2 be any real numbers (not zero) with the same sign, and if all the roots of the equation $\varphi(x)=0$ be real and distinct, then, (1) (1) all the roots of the equation $\varphi(x)-k\varphi''(x)=0$, k being any real number not zero, are real and distinct; (2) there exists no common root between two equations $\varphi(x)-k_1k_2\varphi''(x)=0$ and $\varphi'(x)=0$, and (3) all the roots of the equation $k_1\varphi'^2(x)+k_2\varphi(x)\varphi''(x)=0$ are real and distinct, and are separated by all the roots of the two equations $\varphi'(x)=0$ and $\varphi(x)-k_1k_2\varphi''(x)=0$.

(1) This is evidently a particular case of the above theorem.

Thus, specializing theorems 1 and 2 in several ways, we are able to deduce an infinite number of such results.

3. Theorem 3. *If the roots of the equation $\varphi(x)=0$ of degree n be all real and distinct, and separated by the roots of another equation $\psi(x)=0$ of degree $n-1$, whose roots are all real, then the roots of the equation*

$$\varphi'\psi+k\varphi\psi'=0$$

are all real and distinct, k being any non-negative real constant.

Proof. By the converse of Biehler's theorem, the imaginary parts of all the roots of the equation $\varphi(x)+i\psi(x)=0$ have the same sign (assume positive). Hence those of all the roots of the equation $\varphi(x)+ik\psi(x)=0$ are also positive, k being any positive constant. Hence from Gauss's theorem, it follows that all the points representing the roots of the equation $\varphi'(x)+ik\psi'(x)=0$ are not outside of the least convex polygon which contains all the points representing the roots of the equation $\varphi(x)+ik\psi(x)=0$, and that the imaginary parts of the roots of the equation $\varphi'(x)+ik\psi'(x)=0$ are all positive. Hence if we apply theorem 2 to the product

$$\{\varphi(x)+i\psi(x)\}\{\varphi'(x)+ik\psi'(x)\}\equiv\Phi(x)+i\Psi(x),$$

then we see that all the roots of the equation

$$\Psi(x)\equiv\varphi'\psi+k\varphi\psi'=0$$

are real and distinct, when k is a non-negative real constant.

4. From the last theorem the following problem will arise: Given two equations $\varphi(x)=0$ and $\psi(x)=0$ of degree n_1 and n_2 respectively, and having real roots only, to determine the constants $L_1, L_2, \cdots\cdots, L_m$ for which the equation

$$\varphi^{(m)}\psi+L_1\varphi^{(m-1)}\psi'+L_2\varphi^{(m-2)}\psi''+\cdots\cdots+L_m\varphi\psi^{(m)}=0 \qquad (m\leqq n_1, n_2),$$

have real roots only.

In the following, we will treat a special case of this problem.

Theorem 4. *If an equation $f(x)=0$ of degree n has positive roots only, and if the roots of the equation*

$$x^m+k_1x^{m-1}+k_2x^{m-2}+\cdots\cdots+k_m=0 \qquad (k\leqq n)$$

be all real and negative, then the roots of the equation

$$f+L_1xf'+L_2x^2f''+\cdots\cdots+L_mx^mf^{(m)}=0$$

are all positive, when $L_1, L_2, \cdots\cdots, L_m$ satisfy the relations

$$L_j=\sum_{i=j}^{m}l_i^{(j)}k, \qquad l_i^{(1)}=l_j^{(j)}=1, \qquad l_{i+1}^{(j+1)}=(j+1)l_i^{(j+1)}+l_i^{(j)},$$

$$(j=1, 2, 3, \cdots\cdots, m)\text{(}^1\text{)}.$$

Proof. For any positive number λ_1, the roots of the equation $x^{\lambda_1}f(x)=0$ consist of zeros and the roots of the equation $f(x)=0$. Therefore, differentiating $x^{\lambda_1}f(x)$ with respect to x, it follows by Rolle's theorem that the equation

$$f(x)+\lambda_1^{-1}xf'(x)=0$$

has positive roots only. Hence if we take the second positive number λ_2, the equation

$$(f+\lambda_1^{-1}xf')+\lambda_2^{-1}x(f+\lambda_1^{-1}xf')'=0,$$

i.e. $$f+(\lambda_1^{-1}+\lambda_2^{-1}+\lambda_1^{-1}\lambda_2^{-1})xf'+\lambda_1^{-1}\lambda_2^{-1}x^2f''=0$$

has positive roots only. Therefore, if the roots of the equation

$$x^2+k_1x+k_2=0$$

be all real and negative, the roots of the equation

$$f+(k_1+k_2)xf'+k_2x^2f''=0$$

are all positive.

Further if the roots of the equation

$$x^3+k_1x^2+k_2x+k_3=0$$

be all real and negative, the roots of the equation

$$f+(k_1+k_2+k_3)xf'+(k_2+3k_3)x^2f''+k_3x^3f'''=0$$

are all positive; and so on.

By mathematical induction, we can easily prove that if the roots of the equation

$$x^m+k_1x^{m-1}+k_2x^{m-2}+\cdots\cdots+k_m=0 \qquad (m\leqq n)$$

be all real and negative, the roots of the equation

$$f+L_1xf'+L_2x^2f''+\cdots\cdots+L_mx^mf^{(m)}=0$$

are all positive, where

$$\begin{aligned} L_1&=l_1^{(1)}k_1+l_2^{(1)}k_2+l_3^{(1)}k_3+\cdots\cdots+l_m^{(1)}k_m,\\ L_2&=\qquad\quad l_2^{(2)}k_2+l_3^{(2)}k_3+\cdots\cdots+l_m^{(2)}k_m,\\ L_3&=\qquad\qquad\qquad l_3^{(3)}k_3+\cdots\cdots+l_m^{(3)}k_m,\\ &\cdots\cdots\qquad \cdots\cdots\qquad \cdots\cdots\\ L_m&=\qquad\qquad\qquad\qquad\qquad\qquad l_m^{(m)}k_m, \end{aligned}$$

(1) Since $L_2=0\cdot k_1+1\cdot k_2+3k_3+7k_4+15k_5+31k_6+\cdots\cdot$, the coefficients of k's in L_2 make the sequence $\{2^k-1\}$, $k=0, 1, 2, 3, \cdots\cdot$, which is named the Fermat number-sequence in P. Bachmann's Niedere Zahlentheorie, 2nd part, 1910, p. 75.

and

$$
\begin{array}{ll}
l_i^{(2)}=2l_{i-1}^{(2)}+l_{i-1}^{(1)} & (i=3,4,5,\cdots\cdots,m), \\
l_i^{(3)}=3l_{i-1}^{(3)}+l_{i-1}^{(2)} & (i=4,5,6,\cdots\cdots,m), \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
l_i^{(r)}=rl_{i-1}^{(r)}+l_{i-1}^{(r-1)} & (i=r+1,r+2,\cdots\cdots,m), \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
l_i^{(m-1)}=(m-1)l_{i-1}^{(m-1)}+l_{i-1}^{(m-2)} & (i=m); \\
l_1^{(1)}=l_2^{(1)}=\cdots\cdots=l_m^{(1)}=1, & \\
l_2^{(2)}=l_3^{(3)}=\cdots\cdots=l_m^{(m)}=1; &
\end{array}
$$

that is

$$
L_j=\sum_{i=j}^{m} l_i^{(j)}k_i, \qquad l_i^{(1)}=l_j^{(j)}=1, \qquad l_{i+1}^{(j+1)}=(j+1)l_i^{(j+1)}+l_i^{(j)},
$$

$$
(j=1,2,3,\cdots\cdots,m).
$$

On Some Algebraic Equations Having Real Roots only,

by

TSURUICHI HAYASHI, Sendai.

The theorems and problems in this small contribution were proved, and proposed at a private meeting, a few years ago, and before they were put in complete form for publication Mr. Y. Okada's paper was sent to the Editor, which was inserted in this Journal, Vol. 14, p. 328, and contains the first of them in the last section of his paper. Though all of them, being immediate consequences of De Longchamps-Hermite's theorem, are not so important, I take this opportunity to keep them in record after Mr. Okada's paper.

1. If the roots of the algebraic equation

$$f(x)=a_0+a_1x+a_2x^2+\cdots\cdots+a_nx^n=0$$

be all real, then the roots of the following equations

$$(1)\qquad f(x)+\frac{1}{\lambda_1}(x+k_1)f'(x)=0,$$

$$(2)\qquad f(x)+\left[\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_1\lambda_2}\right)x+\frac{k_1}{\lambda_1}+\frac{k_2}{\lambda_2}\left(1+\frac{1}{\lambda_1}\right)\right]f'(x)$$
$$+\frac{1}{\lambda_1\lambda_2}(x+k_1)(x+k_2)f''(x)=0$$

are all real, λ_1 and λ_2 being positive integers, and k_1 and k_2 being real. For, since the roots of

$$f(x)=0$$

are all real, the roots of

$$(x+k_1)^{\lambda_1}f(x)=0$$

are all real and consequently the equation obtained by differentiation with respect to x, that is

$$\lambda_1(x+k_1)^{\lambda_1-1}f(x)+(x+k_1)^{\lambda_1}f'(x)=0$$

has real roots only. Dividing by $\lambda_1(x+k_1)^{\lambda_1-1}$, we get equation (1). If we apply the same process to the equation (1), considering its left-

hand member as $f(x)$, we arrive at the equation (2). By repetition, we can arrive at a general result.

A particular case of this theorem has been put in a good form by Mr. Okada, loc. cit..

In particular, the roots of the equations

$$f(x)+(x+k_1)f'(x)=0,$$
$$f(x)+(3x+k_1+2k_2)f'(x)+(x+k_1)(x+k_2)f''(x)=0$$

are all real.

If λ become indefinitely great, keeping $\frac{k}{\lambda}$ as finite, then the above equations become those of De Longchamps[1] and Hermite[2].

Now the following question naturally arises: What is the necessary and sufficient condition for the coefficients of the polynomials $P(x)$, that the roots of the equation

$$f(x)+P_1(x)f'(x)+P_2(x)f''(x)+\cdots\cdots+P_k(x)f^{(k)}(x)=0$$

are all real, when the roots of the equation

$$f(x)=a_0+a_1x+a_2x^2+\cdots\cdots+a_nx^n=0$$

are all real, $P_1(x)$, $P_2(x)$,$\cdots\cdots$, $P_k(x)$ being polynomials of degrees 1, 2,$\cdots\cdots$, k, respectively?

I will quote here for reference the following question from H. Laurent's Traité d'Analyse, 3ième partie, 1894, p. 71: "If $f(x)=0$ has real roots only, the equations $f(x)+xf'(x)=0$, and $f(x)+4xf'(x)+x^2f''(x)=0$ have real roots only. Generalize this theorem."

Moreover the following question proposed by De Longchamps and solved by Levavasseur is to be noticed in this connection: "If all the roots of the equation $U=0$ are real, demonstrate that all the roots of the equations

$$f_1=U+(x-a)\,U'=0,$$
$$f_2=U+3\,(x-a)\,U'+(x-a)^2\,U''=0,$$
$$f_3=U+7\,(x-a)\,U'+6\,(x-a)^2\,U''+(x-a)^3\,U'''=0$$

are real, whatever a may be, (a is of course real, and U', U'', U''' are the successive derivatives of U). Give the general expression of $f_n=0$, and demonstrate that, in putting

$$f_n=a_n^0\,U+a_n^1(x-a)\,U'+a_n^2(x-a)^2\,U''+\cdots\cdots,$$

(1) De Longchamps, Journal de Mathématiques Speciales, question No. 278.

(2) Hermite, Nouvelles Annales de Mathématiques, 2 série, t. 5, 1866, p. 479.

we have

$$\begin{aligned} a_n^0 &= 1, \\ a_n^1 &= (2^n-1)/1!, \\ a_n^2 &= (3^n-2\cdot 2^n-1)/2!, \\ a_n^3 &= (4^n-3\cdot 3^n+3\cdot 2^n-1)/3!, \end{aligned}$$

and so on," (Journal de Mathématiques Spéciales de De Longchamps, question No. 54, solved by Levavasseur in 1883, p. 136).

We also find the following two questions in Laurent's Traité, p. 72: (i) "The roots of the equation

$$y=f(x)+af'(x)+a^2f''(x)+\cdots\cdots+a^nf^{(n)}(x)=0,$$

where $f(x)$ is a polynomial of degree n, are all imaginary if those of the equation $f(x)=0$ are all imaginary (make use of the equality $ay'=y-f(x)$). We can conclude that all the roots of

$$f(x)+xf'(x)+x^2f''(x)+\cdots\cdots+x^nf^{(n)}(x)=0$$

are imaginary if those of the equation $f(x)=0$ are so also."

(ii) "If λ denote a positive number and if $\varphi(x)=0$, $\psi(x)=0$ have real roots only, then $\varphi(x)\psi'(x)+\lambda\psi(x)\varphi'(x)=0$ has also real roots only. How must be this theorem modified when the roots of $\varphi=0$ and $\psi=0$ are not all real?"

2. By De Longchamps and Hermite's theorem, we know that if the roots of

$$f(x)=0$$

be all real, the roots of the equation

$$f(x)+k_2f'(x)=0$$

are all real, k_2 being a real constant. Hence by applying this theorem to the equation

$$(x+k_1)f(x)=0,$$

we know that the roots of the equation

$$(x+k_1)f(x)+k_2\{f(x)+(x+k_1)f'(x)\}=0,$$

or

$$(x+k_1+k_2)f(x)+k_2(x+k_1)f'(x)=0$$

are all real. By repetition, the roots of the equation

$$\begin{aligned} &(x+k_3+k_4)\{(x+k_1+k_2)f(x)+k_2(x+k_1)f'(x)\} \\ &\qquad +k_4(x+k_3)\{f(x)+(x+k_1+2k_2)f'(x)+k_2(x+k_1)f''(x)\}=0, \end{aligned}$$

or

$$\{(x+k_1+k_2)(x+k_3+k_4)+k_4(x+k_3)\}f(x)$$
$$+\{k_2(x+k_1)(x+k_3+k_4)+k_4(x+k_3)(x+k_1+2k_2)\}f'(x)$$
$$+k_2k_4(x+k_1)(x+k_3)f''(x)=0$$

are all real.

Herefrom the question naturally arises: What is the necessary and sufficient condition for the coefficients of the polynomials $P(x)$ that the roots of the equation

$$P_k^{(0)}(x)f(x)+P_k^{(1)}(x)f'(x)+P_k^{(2)}(x)f''(x)$$
$$+\cdots\cdots+P_k^{(k)}(x)f^{(k)}(x)=0$$

are all real, when the roots of the equation

$$f(x)=0$$

are all real, $P_k^{(\lambda)}(x)$ being polynomials all of degree k.

3. The necessary and sufficient condition for the sequence k_0, k_1, $k_2,\cdots\cdots$, k_n, that when the roots of the equation

$$a_0+a_1x+a_2x^2+\cdots\cdots+a_mx^m=0$$

are all real, the roots of the equation

$$a_0k_0+a_1k_1x+a_2k_2x^2+\cdots\cdots+a_nk_nx^n=0 \qquad (m\geqq n)$$

are also all real, has been already studied by Profs. J. Schur and G. Pólya in Crelle's Journal, Vol. 144 (1914), pp. 75–113, the first attempt being to be due to Laguerre whose researches are found in his oeuvres, in Nouvelles Annales de Mathématiques, question No. 1392 and 1393, and in Journal de Mathématiques Spéciales de De Longchamps, question Nos. 95 and 123.

From this the question comes forth: What is the necessary and sufficient condition for the sequence $k_0, k_1, k_2,\cdots\cdots, k_n$, that when the roots of the equations

$$(1) \qquad a_0+a_1x+a_2x^2+\cdots\cdots+a_mx^m=0,$$

are all real, the roots of the equation

$$(2) \quad a_0(x+k_0)+a_1(x+k_1)x+a_2(x+k_2)x^2+\cdots\cdots+a_n(x+k_n)x^n=0$$

are all real.

Now let

$$n=m.$$

Then (2) becomes

$$(3) \qquad a_0k_0+a_1k_1x+a_2k_2x^2+\cdots\cdots+a_mk_mx^m+xf(x)=0$$

where

$$f(x)=a_0+a_1x+a_2x^2+\cdots\cdots+a_mx^m.$$

Now when the roots of the equation

$$f(x)=0$$

are all real, those of the equation

$$xf(x)=0$$

are also all real. Hence by the theorem of De Longchamps and Hermite, the roots of

$$xf(x)+k\{f(x)+xf'(x)\}=0$$

are all real. Identifying this equation with (3)

$$\begin{aligned} a_0k_0 &= a_0\,k, \\ a_1k_1 &= 2\,a_1\,k, \\ a_2k_2 &= 3\,a_2\,k, \\ \cdots\cdots & \quad \cdots\cdots \\ a_m\,k_m &= (m+1)\,a_m\,k. \end{aligned}$$

Hence if we choose the sequence $k_0, k_1, k_2, \cdots\cdots, k_m$ as

$$k_0=k, \quad k_1=2k, \quad k_3=3\,k, \cdots\cdots, \quad k_m=(m+1)k,$$

the roots of equation (2) are all real.

This theorem stands in an intimate relation with the following theorem which we find also in Laurent's Traité, p. 72: "If we multiply the successive terms of an equation $\varphi(x)=0$ arranged in ascending or descending order of powers of x by the terms of an arithmetical or geometrical progression, all the roots of the new equation $\psi(x)=0$ are real when all the roots of $\varphi(x)=0$ are real. The same thing takes place if we substitute the series of the form

$$1^m, \ 2^m, \ 3^m, \cdots\cdots$$

instead of the progression, m designating an integer."

Again by my theorem, when the roots of

$$f(x)=0$$

are all real, the roots of

$$xf(x)+\frac{1}{\nu}(x+k)\{f(x)+xf'(x)\}=0$$

are all real. Comparing this equation with the equation

$$a_0(x+k_0)+a_1(x+k_1)+a_2(x+k_2)x^2+\cdots\cdots+a_m(x+k_m)x^m \\ +px^{m+1}=0,$$

which I will take instead of (3), we get

$$
\begin{aligned}
a_0 k_0 &= \nu^{-1} a_0 k, \\
a_1 k_1 &= \nu^{-1} (2a_1 k + a_0), \\
a_2 k_2 &= \nu^{-1} (3a_2 k + 2a_1), \\
a_3 k_3 &= \nu^{-1} (4a_3 k + 3a_2), \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
a_m k_m &= \nu^{-1} \{(m+1) a_m k + m a_{m-1}\}, \\
p &= \nu^{-1}(m+1) a_m ;
\end{aligned}
$$

whence we can determine the values of

$$k_0, k_1, k_2, \cdots\cdots, k_m ; \; p$$

for a given value of k.

July, 1918.

圓錐曲線ニ關スル反轉ニ就テ

On the Inversion with respect to a Conic.

深 澤 清 吾 (群馬縣)

SEIGO FUKAZAWA, Sekine near Maebashi, Gunma-ken.

1. 余ハ玆ニ圓ニ關スル反轉ヲ圓錐曲線ニ關スルモノニ純綜合幾何學的ニ擴張セントス(1). 此ノ問題ニ就キテハ既ニ澤山勇三郎氏ガ東北數學雜誌第 6 卷 (1914–'15), 及ビ東京物理學校雜誌第 299 號 (大正5年9月) 及ビ第 301 號(大正6年12月)ニ於テ説ケル所アリ. 同氏ノ定義ニヨレバ

「一ツノ圓錐曲線内ノ一點 O ヲ過グル弦 MN (必要ナラバ其ノ延長)上ニ於テ二點 P 及 P' ヲ取リ矩形 $OP.OP'$ ガ矩形 $OM.ON$ ノ符號ヲ變ヘタルモノ又ハソレ自身ト等シキ樣ニスルトキハ二點 P ト P' トハ此圓錐曲線ニ關シテ互ニ反形ナリト云フ」.

而シテ同氏ハ積 $OM.ON$ ハ點 O ノ位置ニノミ關スル或量ト OP ノ方向ニノミ關スル或量トノ積ナルコトヲ示サレタリ.

サレバ今 $OM.ON=OL^2$ ナルガ如キ點 L ヲ弦 MN ノ上ニ求メタリトスレバ此點 L ノ軌跡ハ原曲線ト相似應位 homothetic ナル曲線ナリ. 何トナレバ OL ノ長サハ O 點ノ一定セル以上ハ OL ノ方向ニノミ關シ, 之ト平行ナル原曲線ノ半徑ト定比ヲ有スレバナリ. 而シテ點 O ハ此ノ新曲線ノ中心ニシテ二點 P ト P' トハ此ノ新曲線ニ關シテ互ニ反形ナリ.

故ニ澤山氏ノ定義ニ於ケル反形ハ曲線ノ中心ヲ反轉ノ中心トセルモノニ更メ得ベシ, 從ツテ O 點ガ曲線ノ中心ナル場合ニ於テ反形ヲ定義セリトスルモ其ノ一般性ヲ害スルモノニアラズ.

O ヲ曲線ノ中心ナリトスレバ P, M, P', N ハ調和點列ヲナシ P' ハ P ノ極線上ニアリ. 故ニ更ニ次ノ定義ヲ下スコトヲ得.

---

(1) 本問題ハ格段ナル二次反轉 (a quadratic inversion) ナルガ故ニ其一般的研究ニハ歷史多シ. サレド格段ナル場合ニ於ケル研究ノ興味モ亦少ナシトナサズ. 猶ホ此事件ニツキテハ東北數學雜誌第 11 卷 (1917) 第 184 頁ニ於ケル窪田教授ノ小引ヲ參照スルノ要アリ(T. H.)

「一ツノ圓錐曲線 K ノ中心 O ヲ任意ノ一點 P ニ結ブ直線ト P ノ極線トノ交點 P' ヲ曲線 K ニ關スル P ノ反形トイフ.」

然ラバ更ニ之ト同様ニシテ, シカモ O 點ガ K ノ中心ナラザルヤウ, 定義

「一ツノ圓錐曲線 K ト其ノ内ノ一點 O トアリ. 直線 OP ト一點 P ノ K ニ關スル極線トノ交點 P' ヲ P ノ反形トイフ」

ヲ下セバ如何トイフニ, 此ノ定義ハ全然位置的ニシテ射影ニヨリ其反轉關係ニ變化ヲ及ボサズ. 而シテ一點ト一平面トヲ適當ニ選ビテ此ノ點ヨリ全圖形ヲ此ノ平面上ニ射影スレバ O 點ノ射影ヲ新曲線ノ中心タラシムルコトヲ得ベシ. 何トナレバ任意ノ點ヨリ K ヲ射影シテ圓錐ヲ作リ其ノ軸ト O トニテ決定セラルル平面ト圓錐面トノ二交線ヲ求メ, O ヲ通ジテ此ノ二直線間ニ O 點ニテ二等分セラルル直線ヲ作リ, 此ノ直線ヲ通ジテ前記ノ平面ニ垂直ナル平面ニテ圓錐ヲ截斷シ得レバナリ.

此ノ故ニ O 點ガ曲線ノ中心ト一致セザル場合ノ研究ハ, ソノ之レト一致スル場合ノ研究ニ容易ニ歸着セシメ得ベシ. 故ニ余ハ此ノ後ノ場合ニ就キテ研究ノ歩ヲ進メントス. 即チ

「一ツノ圓錐曲線 K ノ中心 O ヲ任意ノ一點 P ニ結ブ直線ト P ノ極線トノ交點ヲ K ニ關スル P ノ反形トイフ」

トノ定義ヲ選擇ス. 但シ抛物線ノ場合ニハ OP ハ其ノ主軸ニ平行ナル直線ナリトシ, 圓錐曲線ガ平行二直線ニ分解シタル場合ニハ中心ハ此ノ二直線ノ中間ニアリテ此ノ二直線ニ平行ナル直線上ノ任意ノ點ナリトス, 又圓錐曲線ガ一致セル二直線トナリタルトキハ反形ハ此直線ニ關スル對稱點ナリ.

2. 吾人ハ既ニ $OP.OP'=OM.ON=OM^2$ ナルコトヲ知レリ. 今 OP ガ K ト交ラザル場合ニハ $OP.OP'$ ハ如何ナル値ヲ取ルカヲ吟味セントス.

OP ガ K ト交ラザル場合ハ雙曲線ニ於テ OP ガ其ノ漸近線ト一致スルカ, 或ハ其ノ共軛雙曲線ト交ルカノ場合ニ於テ起コル. OP ガ漸近線ト一致スル場合ニハ P' ハ無究遠ニアル可ク, $OP.OP'=\infty=OM^2$ ト假定スル事ヲ得ベシ.

次ニ OP ガ K ノ共軛雙曲線ト交ル場合ニハ此ノ交點ヲ M', N' トシ, 此ノ曲線ニ關スル P ノ反形ヲ P'' トス. 然ル時ハヨク知ラレタル

定理「共軛雙曲線ノ各ニ關スル一點ノ極線ハ中心ニ就テ對稱ナリ」ニヨリテ, P' ト P'' トハ O ニ就テ對稱ナルヲ知ル. 從ツテ

$$OP'.OP=-OP''.OP=-OM'^2.$$

今「雙曲線ニハ其ノ共軛雙曲線ト同ジ位置ニアル虛線枝ガ附屬ス」ト假想スル時ハ, 點 OP ガ虛線枝ト交ル點ヲ M トスレバ

$$OM=iOM',$$

$$OM^2=-OM'^2,$$

$$OP.\ OP'=-OM'^2=OM^2.$$

ニシテ此ノ場合ニモ前記ノ關係ハ成立スベシ. 故ニ余ハ理論ヲ一般ナラシムルガ爲ニ上記ノ假想ヲ採用ス.

コノ假想ニヨレバ.

(i) 或雙曲線ノ共軛雙曲線ハ此ノ雙曲線ニ虛單位ヲ乘ゼルモノト考フル事ヲ得ベク. 又

(ii) 或雙曲線ノ共軛雙曲線ニ相似應位ナル曲線ハ原曲線ニモ相似應位ナリト見ルヲ得ベシ.

3. 次ニ余ハ種々ノ圖形ノ反形ヲ求ムルニ先ダチ一種ノ變形(反轉ノ系統全體ヲ變形スルモノ)法ヲ記述セントス.

今 K ニ關シ圖形 R ヲ反轉シテ圖形 Q ヲ得タリトセヨ. O ヲ中心トシ K ヲ縮小(又ハ擴大)シテ K' トナラシメ, 之ニヨリテ R ヲ反轉シテ Q' ヲ得タリトセヨ. O ヲ通ズル一直線ヲ引キ K, K', R, Q, Q' ヲソレゾレ M, M', P, P', P'' ニテ交ラシム. 然ラバ

$$OP.\ OP'=OM^2,$$

$$OP.\ OP''=OM'^2,$$

$$OP' : OP''=OM^2 : OM'^2.$$

此ノ比ノ右邊ハ一定ナルガ故ニ, 左邊モ一定ナラザル可カラズ. 故ニ Q ト Q' トハ O ヲ中心トシテ相似應位ニシテ其ノ相似比ハ K ト K' トノ相似比ノ平方ニ等シ. 此ノ變形法ハ凡テノ曲線ノ形狀ト方向トヲ變ゼザルヲ以テ相似變形ト呼バン. 吾人ハ此ノ變形法ヲ施ス事ニヨリテ

(i) R ト Q' トヲ K' ノ上ニテ交ラシメ,

(ii) 又ハ切セシメ,

(iii) 又ハ一ツヲ全ク K' ノ外ニ一ツヲ全ク內ニ入ラシメ得ベシ. 但 R 又ハ Q' ガ O ヲ通ズル場合ヲ除ク.

4. 吾人ハ之ヨリ種々ノ圖形ノ反形ヲ作リ,而シテ其ノ間ニ介在スル性質ヲ研究セン.

一ツノ曲線ノ反形ヲ作ル最モ一般ノ方法ハ,此ノ曲線上ノ凡テノ點ノ極線ニヨリテ一ツノ線束ヲ作リ, 又中心ヨリ此ノ曲線上ノ凡テノ點ヲ結ブ線束ヲ作リ, 此ノ二ツノ線束ノ產物ヲ研究スルニアリ.

例ヘバ直線ノ反形ヲ作ランニハ,其ノ上ノ點ノ凡テノ極線ハ其ノ直線ノ極ヲ通ズル一次線束ヲ得可ク,又其ノ上ノ凡テノ點ヲ中心ニ結ブ直線ハ前者ト射影的ナル一次線束ヲ作ル可ク,此ノ二線束ノ產物トシテ二次曲線ヲ得ルガ如シ.

5.「直線ノ反形トシテ得タル二次曲線ハ反心 O ヲ通ジ且反轉ノ曲線ニ相似應位ナリ.」

證明. AB ヲ直線トシ, C ヲ其ノ極トス.

OC ヲ K ト交ルトシ. 其ノ交點ヲ M, N トス.

OC ノ中點ヲ O' トス.

AB 上ニ任意ノ一點 P ヲ取リ其ノ反形ヲ P' トス.

CP' ハ P ノ極線ナリ.

ND ヲ OP ニ平行ニ引キ, D ニ於テ K ト交ラシム. 然ル時 MD ハ OP ニテ二等分セラルヽガ故ニ, P 點ノ極線ナル CP' ニ平行ナリ.

三角形 MDN ト $CP'O$ トハ三邊皆平行ナルガ故ニ, OD ト $O'P'$ トハ平行ニシテ定比ヲ有ス,故ニ P' 點ノ軌跡ハ O ヲ通ジ(D ハ N ニ一致シ得ルガ故ニ) K ニ相似應位ナリ.

次ニ特別ノ場合トシテ OC ガ MN ト交ラザル場合ヲ證明セン.

コノ場合ハ雙曲線ニ限ル可ク, 次ノ二ツノ場合アリ.

(i) OC ガ K ノ共軛雙曲線ト交ル場合.

此ノ場合ニハ AB ヲ先ヅ共軛雙曲線ニ就テ反轉シ且中心ニ就テ對稱ナル圖形ヲ求メ,共軛雙曲線ニ相似應位ナル圖形ヲ求ムレバ反形トシテ反轉ノ曲線ノ共軛雙曲線ニ相似應位ナル曲線ヲ得可ク,前記ノ如キ假想ニヨレバ之又反轉ノ曲線ニ相似應位トナシ得可シ.

(ii) AB ガ漸近線ノ一ツニ平行ナル場合, 即チ OC ガ漸近線ト一致スル場合.

C ガ位スル漸近線ヲ EF, 他ノ漸近線ヲ GHトス.

AB ハ EF ニ平行ナル可シ.

AB 上ノ任意ノ一點 P ノ反形ヲ P', CP' ト GH トノ交點ヲ D トス.

CD ハ P ノ極線ナリ.

故ニ CD ノ曲線內ノ部分ハ OP ニテ二等分セラル. 從ツテ漸近線間ノ部分モ又 OP ニテ二等分セラルベシ. 故ニ P' ハ CD ノ中點ナリ. 故ニ其ノ軌跡ハ GH ニ平行ナル直線ナリ.

此ノ場合ニハ反形ハ反心ヲ通ゼザルモ, 反心ヲ通ジ反轉ノ曲線ニ相似應位ナル圖形ノ極限ト考フルヲ得ベシ.

カクノ如ク考フル時ニ於テ此ノ定理ハ一般ニ成立ス可シ.

抛物線ニ於テハ O 及 N ガ無窮遠ニアリト假想シテ同樣ニ證明シ得ベシ.

6. 「反心ヲ通ジ反轉ノ曲線ニ相似應位ナル曲線ノ反形ハ直線ナリ.」

證明. O ヲ通ジ K ニ相似應位ナル曲線ヲ R トス.

R ノ上ノ任意ノ二點 P', Q' ノ反形ヲ P, Q トセヨ.

P, Q ヲ結ブ直線 AB ノ反形ヲ作レバ此ノ反形ハ P', Q', O ナル三點ヲ通ジ K ニ相似應位ナルガ故ニ R ニ外ナラズ.

故ニ AB ノ反形ハ R ナリ. 故ニ R ノ反形ハ AB ナルベシ.

吾人ハコヽニ偶然ニシテ「反心ヲ通ジ」「反轉ノ曲線ニ相似應位ナル」曲線ノ反形ヲ得タリ. 之ヨリ此ノ二ツノ條件ノ內任意ノ一ツヲ除キテ

「反心ヲ通ズル二次曲線ノ反形如何」

「反轉ノ曲線ニ相似應位ナル曲線ノ反形如何」

ヲ研究スルハ當然ノ順序ナルベシト信ズ. 次ニ之ニ就キ答フル所アラントス.

7. 「反心ヲ通ズル任意圓錐曲線ノ反形ハ一般ニ三次曲線ナリ.」

此ノ定理ハ一般ノ方法ニヨリ證明スルヲ得ベシ. 卽二次點列ノ極線ハ之ニ射影的ナル二次線束ヲ作ルベク, 之ノ中心ヲ心トシ之ト射影的ナル一次線束トノ產物トシテ三次曲線ヲ得可ケレバ也.

余ハコヽニ此三次曲線ノ性質ハ暫ク措キ此ノ三次曲線ガ二次曲線ト一次曲線又ハ三個ノ直線ニ分解スル場合ヲ吟味セン. 但シ三個ノ直線ニ分解スルハ, 前定理ニヨリ此ノ二次曲線ガ反轉ノ曲線ニ相似應位ナ

ルトキナルヲ知ル. 故ニ今此三次曲線ガ二次曲線ト一次曲線トニ分解スル場合ノミヲ論ズベシ.

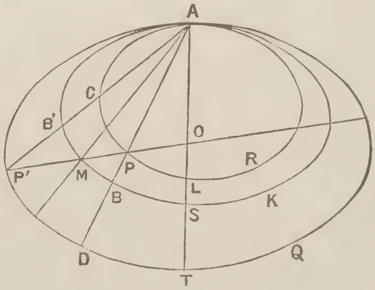
抑々三次曲線ガ二次曲線ト一次曲線トニ分解スル爲ニハ其ノ母形タル一次線束ト二次線束トガ一個ノ對應元素ヲ共有セザル可カラズ. 即チ中心ヲ通ズル二次曲線 R ノ反形ガ二次曲線ト直線トニ分解スル爲ニハ R ノ上ノ凡テノ點ヲ反心ニ結ブ一次線束ト R ノ上ノ凡テノ點ノ極線トヨリ成ル二次線束トガ一個ノ對應元素ヲ共有セザル可カラズ. 即チ R ノ上ニ一點 P アリテ其ノ極線ガ P ト反心 (O トス) トヲ結ブ直線ト一致スルガ如クナラザル可カラズ.

今斯ノ如キ點 P ヲ求メントス.

P ノ極線ハ PO ノ共軛直徑ニ平行ナリ. 之ガ PO ト一致スルガ爲ニハ, 反轉ノ曲線 K ハ雙曲線ニテ, PO ハ其ノ漸近線ノ一ツナラザル可カラズ. 而シテ K ノ漸近線上ノ總テノ點ノ極線ヲ作リ試ミレバ, 無窮遠ニ於ケル曲線トノ交點ニ於テノミ, PO ト P ノ極線ト一致スルヲ見ル可シ. 即チカクノ如キ點 P ハ K ノ漸近點トノ交點ノミナル可ク, カクノ如キ唯一點ヲ有スル曲線 R ハ K ト唯一ツノ漸近線ノ方向ヲ共有スル曲線ナル可シ. 而シテ此ノ雙曲線ノ反形ヲ求ムレバ, ソハ O ニ於ケル R ノ切線ト, R ノ漸近線ニ平行ナラザル K ノ漸近線トニ平行ナル漸近線ヲ有シ, 且 O ヲ通ズル雙曲線ナルベシ.

而シテ三次曲線ノ第三枝(直線ナル)極限ハ R ノ漸近線ニ平行ナル K ノ漸近線ナルヲ見ルベシ. 故ニ「反轉ノ曲線 K ノ中心ヲ通ジ K ト一ツノ漸近線ノ方向ヲ共有スル雙曲線ノ反形ハ亦 K ノ中心ヲ通ジ他ノ漸近線ノ方向ヲ共有スル雙曲線ナリ」ト云ヒ得ベシ. 而シテ此ノ場合, 及前定理ノ場合ノ外ハ凡テ「中心ヲ通ズル二次曲線ノ反形ハ三次曲線ナル」ヲ見ルベシ.

8. 「反轉ノ曲線ニ相似應位ナル曲線ノ反形ハ又之ニ相似應位ナル曲線ナリ.」



證明. K ヲ反轉ノ曲線トシ R ヲ之ニ相似應位ナル曲線トス.

相似變形法ニヨリ K ト R ト相切セシム. 此ノ點ヲ A トス.

A, O ヲ結ビ R ヲ L ニテ, K ト S ニテ交ラシメ,

A, L, S ノ第四調和點 T ヲ求メ,

A ニ於テ K 及 R ニ切シ T ヲ通ジ且 K ニ相似應位ナル曲線 Q ヲ作ル.

然ル時ハ Q ガ R ノ反形ナル事ヲ證明セントス.

O ヲ通ジ任意ノ直線ヲ引キ K ヲ M 及 N ニテ, R 及 Q ト P 及 P' ニテ交ラシムベシ.

A, P' ヲ結ビ R, K ト B', C ニテ交ラシメ,

AP ヲ結ビ K, Q ト B, D ニテ交ラシム.

A, C, B', P' ノ各點ノ距離ハ A, L, S, T ノソレニ比例スルガ故ニ A, C, B', P' ハ調和點列ナリ.

PC ト BB' トハ平行ナリ.

$P(A, C, B', P')$ ナル調和線束ノ一線 PC ニ平行ナル BB' ハ PP' ニテ二等分セラレザル可カラズ.

BB' ハ OP ニテ二等分セラルヽガ故ニ $B'MDN$ ハ調和系ヲナス.

故ニ之ヲ A ヨリ射影シテ OM ニテ截斷シタル $P'MPN$ モ亦調和點列ヲナス.

故ニ P' ハ P ノ反形ナリ.

故ニ Q ハ R ノ反形ナリ.

9. 余ハ之ヨリ前定理ノ最モ一般ナル逆

「圓錐曲線 R ガ圓錐曲線 K ニヨリ圓錐曲線 Q ニ反轉スル時 R, K, Q ハ相似應位ナリ」

トノ定理ヲ證明セントス. 但シ Q, R ノ各々ハ K ノ中心ヲ通ゼザルモノトス. Q, R ノ或一ツガ K ノ中心ヲ通ズル場合ニハ他ノ一ツハ三次曲線トナル可ク. 之ガ二次曲線ニ分解スル場合ニハ本定理ハ必ズシモ成立セザル事前ニ見タルガ如シ.

此ノ定理ハ反轉論應用ノ方向ト限界トヲ指示スル極メテ重要ナルモノナリ. 今之ヲ次ノ二個ノ補題ニヨリテ證明セン.

「R ト Q トガ相似應位ナレバ K モ亦之ニ相似應位ナリ」

「K ノ中心ノ R 及 Q ニ關スル極線ハ平行ナリ.」

補題ノ前者モ亦重要ニシテ興味アルモノナレド其ノ證明ヤヽ複雜ナリ.

10. 補題第一ノ證明.

(i) 曲線 K, Q, R ノ中心ハ一直線上ニアリ.

K ノ中心ハ R, Q ノ双方ノ内ニアルカ双方ノ外ニアルカナリ. 何トナレバ K ノ中心ヨリ R ヘ引ケル切線ハ又 Q ニモ切ス可ケレバ也.

後者ニアリテハ K ノ中心ハ R, Q ノ共通切線ノ交點ナルガ故ニ R, Q ノ中心ト一直線上ニ在リ.

前者ニアリテハ, K ノ中心ヲ通ジ且之ニテ二等分セラルヽ R ノ弦ヲ引ケバ, 此ノ直線ノ Q 内ノ部分モ亦 O 點ニテ等分セラル可ク, 此直線ヲ a トス.

R ノ中心ト O トヲ結ブ直線ハ R ニ於テ a ニ平行ナル直徑ノ共軛直徑ナルベク, 又 R ノ中心ト O トヲ結ブ直線ハ Q ニ於テ a ニ平行ナル直徑ナルベシ.

R ト Q トハ相似應位ナルガ故ニ, a ニ平行ナル直徑ノ共軛直徑モ亦平行ナラザルヲ得ズ.

然ルニ此等ハ O 點ヲ共有ス.

故ニ一致セザルヲ得ズ.

故ニ O ト R ノ中心ト Q ノ中心トハ一直線上ニアリ.

何レニシテモ O ハ R ト Q トノ中心ト一直線上ニアリ.

(ii) K ノ中心ハ一般ニ求メ得可シ.

R ノ中心ト Q ノ中心トヲ結ブ直線ガ R ト交ル點ヲ M, N トシ Q ト交ル點ヲ M', N' トス. 又 K ト交ル點ヲ L, L' トス.

L, L' ハ M, M' ト N, N' 又ハ M, N' ト M', N ナル點對ヲ共ニ調和ニ分ツ點對ナリ.

而シテ O ハ LL' ノ中點ナルガ故ニ O ノ位置ハ只 M, N, M', N' ナル四點ノ位置ノミニ關ス.

M, N, M', N' ナル四點ノ配合ニハ次ノ三種アリ.

(イ) M, N, M', N'. 此ノ場合ニハ MN 及 $M'N'$ ヲ直徑トスル相似應位ナル楕圓ヲ畫キ其ノ共通切線ノ交點ヲ求ムレバ之即 O 點ナリ.

(ロ) M, M', N, N'. 此ノ場合ニハ MM', NN' 及 $MN', M'N$ ナル二組ノ點對ハ共ニ雙曲線的ナルガ故ニ, 二組ノ L, L' ヲ求メ得ベシ. 從ツテ二箇ノ O 點ヲ求メ得可シ.

(ハ) M, M', N', N. 此ノ場合ニハ (イ) ノ場合ノ代リニ雙曲線ヲ用ヒテヤハリ二個ノ O 點ヲ求メ得ベシ.

要スルニ O 點ハ何レノ場合ニ於テモ二個求メ得可シ.

(iii) 結論. スデニ反心ヲ求メ得タリ, 之ニヨリテ相似變形ヲ施シ得可シ.

對應點ガ中心ニ就テ反對ノ側ニアル時ハ K ニ虛單位ヲ乘ジタリト考ヘテ, 之ヲ同ジ側ニ移シ得ベシ. カクシテ Q ト R トヲ相切セシム. 然ル後第8節ノ方法ニ依ツテ K ヲ作圖スル時ハ之ガ反轉ノ曲線ニ外ナラザルコト同節ノ論法ヲ逆ニシテ證明シ得ベシ.

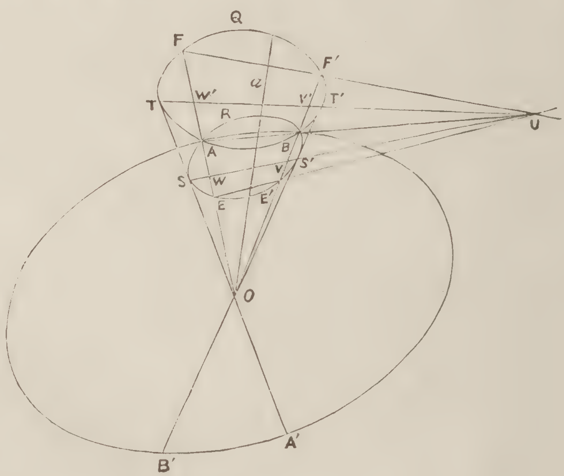
故ニ何レニシテモ K ハ Q, R ニ相似應位ナル曲線ニシテ二個存在ス.

但シ其ノ內一ツハ虛線ナル事アリ.

11. 補題第二ノ證明.

(i) O ガ R, G ノ双方ノ內部ニアル時.

前節 (i) ニ於ケルト同樣ノ論法ニヨリ, O ヲ通ジ此ノ點ニテ二等分セラルル R ノ弦 a ヲ作レバ, a ノ Q 內ノ部分モ亦 O 點ニテ二等分セラル可ク, a ハ O 點ノ R ニ關スル極線ニモ, 又 Q ニ關スル極線ニモ平行ナリ.



故ニ O 點ノ R 及 Q ニ關スル極線ハ相互ニ平行ナラザル可カラズ.

(ii) O ガ R 及 Q ノ双方ノ外部ニアル時.

Q ト R トガ K ノ上ニテ交ルトス. (相似變形ニヨリカクナサシム.) 此ノ交點ヲ A, B トス. QR ノ共通切線ヲ $ST, S'T'$ トスレバ SS', TT' ガ平行ナル事ヲ證ス可キナリ, SS' ト TT' トノ交點ヲ U トス. U ノ R 及 Q ニ關スル極線ハ一致ス. 之ヲ a トス.

a ト UB トノ交點ヲ U' トスレバ U, B, U' ノ第四調和點ハ R 上ニモ, 又 Q 上ニモアル可ク, 從ツテ此ノ點ハ A ニ外ナラズ.

故ニ AB ハ U ヲ通ズ.

OA 及 OB ト Q 及 R トノ第二交點ヲソレゾレ, $E, F; E', F'$ トシ又, SS', TT' トノ交點ヲ $W, W'; V, V'$ トス.

又 K トノ第二交點ヲ A', B' トス.

次ノ各組ノ點列ハ配景的調和點列ナリ.

(イ)　　$OE'VB,\quad OEWA.$

(ロ)　　$OBV'F',\quad OAW'F.$

之ニヨリテ $EE', WV, AB, W'V', FF'$ ハ皆 U ニ於テ交ル.

二ツノ調和點列 $A'EAF, B'E'BF'$ ノ三双ノ對應點ノ交線 EE', AB, FF' ハ U ニ於テ交ルガ故ニ, $A'B'$ モ亦 U ヲ通ズ.

U ハ $A'B'$ ト AB トノ交點ナルガ故ニ無窮遠點ナリ. 故ニ SS' ト TT' ハ平行ナリ.

12. 「圓錐曲線 K, Q, R, アリ. K ニ關シ Q, R ガ互ニ反形ナル時 K, Q, R ハ共ニ相似應位ナリ」.

證明. O ノ R 及 Q ニ關スル極線ヲ a, b トス.

O ヲ通ズル任意直線ヲ引キ Q, R, K, a, b ト $S, P; P', S'; M, T, T'$ ニテ交ラシム.

然ルトキハ

$$OS.\,OS' = OM^2 = OP.\,OP',$$

$$OS : OP' = OP : OS' = OT : OT' = \text{一定}$$

故ニ R, Q ハ相似應位ナリ.

故ニ K, Q, R ハ共ニ相似應位ナリ.

抄　録　短　評

I. 新 刊 書 目

E. H, Askwith. A course of pure geometry containing a complete geometrical treatment of the properties of the conic sections. New edition revised and enlarged. Cambridge, University Press, 1917. 11+284 p. 7s. 6d.

W. W. R. Ball. Cambridge papers. London, Macmillan, 1918. 6+326 p. 6s.

N. G. W. H. Beeger. Over eene functie, voregesteld door eene reeks van Dirichlet. Proefschrift. Groningen, Gebr. Hoitsema, 1916. 84 p.

Berichte und Mitteilungen veranlasst durch die Internationale Mathematische Unterrichtskommission. 2te Folge III: Titel und Inhaltsübersicht zur ersten und zweiten Folge der Berichte und Mitteilungen. Leipzig, Teubner, 1917. 16+99 p. M. 4.00.

L. von Bortkiéwicz. Die Iterationen. Ein Beitrag zur Wahrscheinlichkeitstheorie. Berlin, Springer, 1917. 12+206 p. Geb. M. 10.00.

W. C. Brenke. Advanced algebra. New York, Century Company, 1917. 8+196 p. $1.25.

W. C. Brenke. Elements of trigonometry with tables. New York, Century Company, 1917. 6+122+40 p. $1.25.

F. S. Carey. Infinitesimal calculus. Section 2. London, Longmans, Green and Company, 1917.

U. Cisotti. Lezioni di analisi matematica, svolte nel r. Istituto tecnico superiore di Milano. Pavia, tip. Cooperativa, 1918. 16+674 p.

E. S. Crawley and H. B. Evans. Analytic geometry. Philadelphia, E. S. Crawley, 1918. 14+239 p.

P. Gauja. Les fondations de l'Académie des Sciences (1881–1915), avec une préface par A. Lacroix et É. Picard, secrétaires perpetuels de l'Académie des Sciences. Paris, Gauthier-Villars, 1917. 6+658 p. Fr. 12.00.

C. Huygens. Oeuvres complètes de Christian Huygens publiées par la Société Hollandaise des Sciences. 13 tomes in 14. La Haye, Nijhoff, 1888–1916. Couv. vélin. Fl. 180.00.

International catalogue of scientific literature. Fourteenth annual issue. A. Mathematics. London, Harrison, 1917. 8+163 p. 15 s.

G. Javelot. Comment résoudre les problèmes de géométrie élémentaire. Paris, F. Nathan, 1918. Fr. 3,00.

L. Koenigsberger. Weierstrass' erste Vorlesung über die Theorie der elliptischen Funktionen. Leipzig, Teubner, 1917. 32 p.

C. Michel. Cours d'algèbre et l'analyse Paris, Alcan, 1916. 10+860 p.

R. M. Milne. Mathematical papers for admission into the Royal Military Academy and the Royal Military College for the years 1908–1917. Edited by R. M. Milne. London, Macmillan, 1918. 7s.

M. d'Ocagne. Cours de géométrie pure et appliquée de l'Ecole polytechnique. Tome 2: Cinématique appliquée. Stéréotomie. Statique graphique. Calcul grafo-mécanique. Nomographie. Paris, Gauthier-Villars, 1918. 6+364 p. Fr. 18.00.

A. Pringsheim. Vorlesungen über Zahlen-und Funktionenlehre. 1ter Bd. 2te Abteilung: Unendliche Reihen mit reellen Gliedern. Leipzig, Teubner, 1916. 8+222 p. M. 10.80.

B. Russell. Mysticism and logic and other essays. New York, Longmans, Green and Company, 1918. 8+234. $2.50.

J. B Shaw. Lectures on the philosophy of mathematics. Chicago and London, Open Court, 1918. 8+206 p. $1.50.

G. Stadler. Études sur l'équation $\delta^p u/\delta x^p - a\Delta^m u = 0$. (Diss., Lund.) Lund, 1916. 6+89 p.

M. Viscardini. Nota su alcune corrispondenze duali di carattere proiettivo, metrico e statico sulla sfera e sul piano. Roma, Stabilimento tipolitografico del Genio Civile, 1918. 12 p. 2 tavole.

藤澤利喜太郎講義, 確率及統計論參考書類(獨逸文). 非賣品

遠藤利貞遺著, 增修日本數學史. 東京帝國學士院藏版, 岩波書店發行. 大正 7 年 1918 702 頁, 4 圓.

故遠藤利貞氏ガ拮据黽勉日本數學史ノ大著ヲ修シ之ヲ公ケニセシハ明治 29 年 1896 ノコトナリシガ當時ノ版ハ四號活字ニテ 450 頁ニ充タザリシニ同氏ガ其後チ世ヲ終フルマデニ次第ニ加筆增修セシ草稿餘程ノ大冊トナリ, 帝國學士院, 特ニ故菊池男爵ノ推奬, ニヨリ此度ハ五號活字ニテ 702 頁トナリテ出版セラル. 前版既ニ斯道研究ノ羅針盤トシテ定評アルモノ, 今ハ益々材料ヲ增加セラレテ更ニ完璧トナレルニ近シ. 遠藤氏ノ畢生ノ大事業トシテ余等ハ此ノ新著ノ有難味ヲ特ニ感受スルモノナルガ, 本邦人ノ數理上ノ貢献ガ如何ニ大ナリシカヲ知ラントスルモノハ本書ニ依リテ其道ヲ拓クヲ得ベク, 又本邦人ノ科學ニ對スル頭腦ノ封建時代ニ於テ既ニ卓越セルモノアリシヲ知ラン. 增修出版ニ就キテハ菊池男爵及三井男爵ノ盡力勿論大ナリト雖モ稿本整理ノ任ニ當レル三上義夫君ノ勞大ニ感謝スベク岡本則錄氏及ビ大谷亮吉氏ノ補添亦有益ナリ. 特ニ卷末ノ人名及ビ書名索引ハ便利ヲ與フルコト大ナリ.

菊池大麓著, 普通幾何學大要. 東京, 大日本圖書株式會社發行. 大正 7 年 1918. 381 頁 2 圓 70 錢.

故菊池男爵ガ我邦ノ普通幾何學ノ教授ニ熱心ニシテ大ナル良績ヲ擧ゲ得タルモノ世人ノ十分ニ熟知セル所ナリ. 今又其ノ薨去後ニ於テ遺著トシテ本書ノ出版アリ. 序文無用ヲ主張セラルル藤澤博士ガ特ニ空前ニシテ多分絕後ナルベシトセラルヽ序文ヲ書カヽレテ世ニ推奬セラルルヲ以テスルモ其ノ好著タルコト明白ナリ. 故男爵ノ恩惠ニ浴スル者本書ヲ繙キテ更ニ其ノ功德ヲ偲ブノ情ノ切ナルモノアリ. 本書ノ如キ好著ノ廣ク布カレテ普通幾何學界ノ敎導トナランコトハ固ヨリ燎然タリ.

藤田外次郎, 刈屋他人次郎, 梶島二郎合編, 數學公式, 附要項及ビ諸表. 東京數學專修學會藏版, 山海堂出版部發行. 大正 7 年 1918. 257 頁. 90 錢.

數學智識ノ擴汎ノ頗ル大ナルモノアル今ノ時ニアリテ本書ノ如キ袖珍的公式集及ビ諸表ノ出版ハ極メテ當ヲ得タルモノトイフベシ.

林鶴一, 西村秀雄共著, 射影幾何學. 東京, 大倉書店, 大正 7 年 1918. 579 頁. 3 30 圓.

本書ハ數叢學書第 23 篇トシテ發行セラル. 射影幾何學ノ要領ヲ會得セントスル人ニトリテ良教科書ナリトスベシ. 述ブル所平面上ノ一次圖形及ビ二次圖形ニ限ラレタリト雖モ先ヅ此程度ニ限リテ滿足スベシトナス. シカモ極メテ多ク圖版ヲ費用ヲ吝マズ挿入シ且ツ問題ハ重要ナルモノヲ選ビテ悉ク解說ヲ施コセリ. 此ノ約 600 頁ニ垂ントスル大册亦我數學界進步ノ一助タルコトヲ期待スルモノナリ. 普通幾何學ヲ終ヘタル人々ガ解析幾何學ニノミ走ラントスルヲ止メテ此ノ華麗ナル射影幾何學ヲ賞玩シ眞ノ幾何學味ヲ感得スルニハ一便宜ヲ得タルモノトイフベシ.

II. 雑誌内容

下記ノ雑誌ニ掲載セラレタル論文中，數學又ハ數理物理學ニ關係ナキモノハ省略ス.

The Science Reports of the Tôhoku Imperial University, 1st Ser., Vol. 7, No. 1, July, 1918.

T. Hayashi, On a certain functional equation. T. Kubota, Über die (2, 2)-deutigen quadratischen Verwandtschaften, II.

Memoirs of the College of Science of the Kyôto Imperial University, Vol. 3, No. 5, June, 1918.

M. Sono, On congruences, II.

Tôkyô Sûgaku-Buturigakkwai Kizi (Proceedings of the Tôkyô Math.—Physical Society), 2nd Ser., Vol. 9, No. 16–19, May-Sept, 1918.

J. Ishiwara, Eine relativistische Theorie der Gravitation von skalarem Potential K. Ogura, Note supplementary to the paper "On the Striped net of curves without ambages in dynamics".

Proceedings of the London Mathematical Society, (2), Vol. 16, Part 5, April, 1918.

L. J. Rogers, On two theorems of combinatory analysis and some allied identities (continued). W. H. Young and G. C. Young, On the internal structure of a set of points in any number of dimensions. P. A. MacMahon, Small contribution to combinatory analysis.

The Messenger of Mathematics, Vol. 47, No. 8–10, Dec, 1917–Feb., 1918.

G. N. Watson, A theorem of Liouville's type (continued). C. E. Weatherburn, Green's dyadics in the theory of elasticity. C. V. Hanumanta Rao, On a certain definite integral. L. J. Mordell, The class number for definite binary quadratics. On the solutions of $x^2+y^2+z^2+t^2=4m_1m_2$. G. H. Hardy, Notes on some points in the integral calculus (48). G. N. Watson, Approximate formulae for Legendre functions.

The Mathematical Gazette, Vol. 9, No. 134-5, March–May, 1918.

W. P. Milne, The graphical treatment of power series. S. Brodetsky, The elementary theory of statical stability. W. J. Dobbs, The introduction to infinite series.

Annals of Mathematics, (2), Vol. 19, No. 4, June, 1918.

T. Fort, A class of developments in orthogonal functions. W. G. Simon, A formula of polynomial interpolation. G. M. Green, Plane nets with equal invariants. D. R. Curtiss, Recent extensions of Descartes' rule of signs. P. J. Daniell, A general form of integral. T. H. Gronwall, Elastic stresses in an infinite solid with a spherical cavity.

Bulletin of the American Mathematical Society, Vol 24, No. 8–10, May–July, 1918.

F. N. Cole, The february meeting of the American Mathematical Society. E. T. Bell, Some remarkable determinants of integers. H. Blumberg, A theorem on semicontinuous functions. F. H. Safford, Surfaces of revolution in the theory of Lamé's products. C. J. Keyser, Note concerning the number of possible interpretations of any system of postulates. L. P. Eisenhart, Darboux's Principles of Geometry. B. A. Bernstein, The April meeting of the San Francisco Section. A. Dresden, The April

meeting of the American Mathematical Society at Chicago. P. R. Rider, A theorem on the variation of a function. C. H. Forsyth, Tangential interpolation of ordinates among areas. R. C. Archibald, Non-Euclidean geometry. R. D. Carmichael, Functions of two complex variables. F. N. Cole, The April meeting of the American Mathematical Society in New York. W. C. Graustein, Note on isogenous complex functions of curves. A. A. Bennett, An elementary derivation of the probability function. G. H. Light, The intrinsic equation for Euler's resistance integral. J. Pierpont, Hermite's Works. H. Hancock, Remarks on elliptic integrals.

Transactions of the American Mathematical Society, Vol. 19, No 2, April, 1918.

H. H. Mitchell, Proof that certain ideals in a cyclotomic realm are principal ideals. W. A. Manning, The order of primitive groups (III). W. E. Milne, On the degree of convergence of Birkhoff's series. C. E. Wilder, Problems in the theory of ordinary linear differential equations with auxiliary conditions at more than two points. L. P. Eisenhart, Transformations of applicable conjugate nets of curves on surfaces. R. W. Brink, A new integral test for the convergence and divergence of infinite series. W. D. MacMillan, On the reduction of certain differential equations of the second order. E. F. Simonds, Invariants of differential configurations in the plane.

American Journal of Mathematics, Vol. 40, No 3, July, 1918.

O. D. Kellogg, Interpolation properties of orthogonal sets of solutions of differential equations. H. B. Phillips, Directed integration. Lepine Hall Rice, P-way determinants, with an application to transvectants. W. Harold Wilson, On a certain general class of functional equations. R. G. D. Richardson, Contributions to the study of oscillation properties of the solutions of linear differential equations of the second order

The American mathematical Monthly, Vol. 25, No. 4–6, April–June, 1918.

W. H. Roever, Descriptive geometry and its merits as a collegiate as well as an engineering subject. E. B. Stouffer, Geometry for juniors and seniors. F. Cajori, Origin of the name "mathematical induction". A. J. Kempner, Miscellanea. N. Altshiller, On the I-centers of a triangle. K. P. Williams, Note on continuous functions. E. W. Chittenden, Note on functions which approach a limit at every point of an interval. H. N. Wright, The nine-point circle obtained by methods of projective geometry.

Proceeding of the National Academy of Sciences of the United States of America, Vol. 4, No. 3–7, March–July, 1918.

O. E. Glenn, Invariants which are functions of parameters of the transformation. G. H. Hardy, On the representation of a number as the sum of any number of squares, and in particular of five or seven.

The Journal of Indian Mathematical Society, Vol. 10, No. 2, April, 1918.

M. T. Naraniengar, Complex roots of equations. G. A. Miller, The function of mathematics in scientific research.

Journal de Mathématiques pures et appliquées, (7), t. 3, fasc. 1.

P. Montel, Sur la representation conforme. J. A. Barrau, Mouvements algébriques dans le plan.

Annales Scientifiques de l'École Normal Supérieure, (3), t. 35, No. 1–2, Jan.–Fév., 1918.

J. Boussinesq, Complément à un récent mémoire des Annales scientifiques de l'École Normal Superieure, sur la poussée des terres et l'état ébouleux, avec quelques idées générales sur la mécanique des semi-fluides et application de ces idées aux corps plastiques.

Bulletin de la Société Mathématique de France, t. 45, Fasc. 4, 1917.

E. Cartan, La déformation des hypersurfaces dans l'espace à $n \geqq 5$ dimensions (suite et fin). E. Cotton, Sur l'abscisse de convergence des série de Dirichlet. W. Sierpiński, Démonstration élémentaire d'un théorème de M. Borel sur les nombres absolument normaux et détermination effective d'un tel nombre. H. Lebesgue, Sur certaines démonstrations d'existence. T. Lalesco, Les classes de noyaux symétrisables. P. Apell, Sur des polynomes se rattachant à l'équation différentielle $y'' = 6y^2 + x$. G. Valiron, Sur les chemins de détermination des fonctions entières.

Nouvelles Annales de Mathématiques, (4) t. 18, Mars-Mai, 1918.

R. Goormaghtigh, Sur l'affinité imaginaire. A Myller, Surfaces parallèles aux surfaces cyclides. M. Weill, Théorèmes généraux sur des systèmes de courbes et points. J. Arnovlievitch, Sur les théorèmes des projections et des moments des quantités de mouvement. R. Goormaghtigh, Sur les faisceaux de coniques. R. Bouvaist, Note de géometrie infinitésimale. F. Balitrand, Sur les cubiques circulaires. Auric, Contribution à la résolution géométrique de l'équation du troisième degré. J. B. Pomey, Intégration de l'équation différentielle linéaire à coefficients constants.

L'enseignement Mathématique, Année 19, No. 4–6, Juillet–Nov., 1917, Anée 20, No. 1, Janv. 1918.

E Dumont, Décomposition des segments de droites en parties égales. F. Daniels, Note sur la géométrie du triangle et du tétraèdre. L. Braude, Sur une généralisation du théorème de Steiner-Habich concernant les roulettes et les podaires appliquée aux orbiformes d'Euler. A. Aubry, Note sur les permutation (Définitions, classifications et transformations, I). H. E. Hansen, L'équation de Fermat $a^{p-1} = pk_p(a) + 1$. F. Gonseth, Sur une transformation projective conduisant à quelques propriétés métriques. G. Tiercy, Sur la fonction résistance $F(v)$ de la balistique.

C. De La Vallée Poussin, L'approximation des fonctions d'une variable réelle. A. Buhl, Deux récents ouvrages de géométrie. G. Loria, Remarque sur la construction des courbes gauches avec application à la parabole cubique. M. Zack, Théorie élémentaire de la toupie gyroscopique.

Comptes Rendus des Séances de l'Académie des Sciences de Paris, (2) t. 165, No. 20–26, t. 166, No. 1–20.

M. Fréchet, Les fonctions prolongeables C. Humbert, Sur le développement, en fraction continue de Stephen Smith, des irrationnelles quadratiques. W. H. Young, Sur les séries des polynomes de Legendre. P. Humbert, Réduction de l'équation des jacobiens critiques. C. Guichard, Sur les réseaux C tels que l'équation de Laplace qui y correspond soit intégrable. P. Humbert, Expression de la fonction de Legendre de seconde espèce. É. Picard, Sur une équation fonctionnelle se présentant dans la théorie de la distribution de l'électricité avec la loi de Neumann. W. De Tannenberg, Sur une question d'analyse indéterminée. P. Fatou, Sur les substitutions rationnelles. G. H. Hardy, J. E. Littlewood, Sur la convergence des séries de Fourier et des séries de

Taylor. G. Julia, Sur les substitutions rationnelles. Akimoff, Transcendantes de Fourier-Bessel à plusieurs variables.

G. Giraud, Sur les fonctions hyperabéliennes. S. Lattès, Sur l'itération des substitutions rationnelles et les fonctions de Poincaré. J. Chokhate, Sur quelques propriétés des polynomes de Tchebicheff. A. Denjoy, Sur une propriété générale des fonctions analytiques. G. Julia, Sur l'itération des fractions rationnelles. M. D'Ocagne, Sur les surfaces gauches circonscrites à une surface donnée le long d'une courbe donnée. R. Garnier, Sur les singularités irrégulières des équations différentielles linéaires. L. Bloch, Sur les théories de la gravitation. A. Buhl, Sur certaines somme abéliennes d'intégrales doubles. S. Lattès, Sur l'itération des substitutions rationnelles à deux variables. G. Julia, Sur des problèmes concernant l'itération des fractions rationelles. F. Iversen, Sur les valeurs asymptotiques des fonctions méromorphes et les singularités transcendantes de leurs inverses. P. Barbarin, Sur le dilemme de J. Bolyai. P. Fatou, Sur les équations fonctionnelles et les propriétés de certaines frontières. A. Denjoy, Sur les courbes de M. Jordan. D. Pompiu, Sur une définition des fonctions holomorphes. R. De Montessus De Ballore, Sur les quadratiques gauches de première espèce. A. Buhl, Sur la représentation, par des volumes, de certaines sommes abéliennes d'intégrales doubles. P. E. Gau, Sur l'intégration des équations aux dérivées partielles du second ordre. M. T. Beritch, Extension du théorème de Rolle au cas de plusieurs variables. B. Jeknowski, Généralisation d'un théorème de Cauchy relatif aux développments en séries. R. De Montessus De Ballore, Sur les quadratiques gauches de première espèce. J. Ritt, Sur l'itération des fonctions rationnelles. Valiron, Démonstration de l'existence pour les fonctions entières, de chemins de détermination infinie. T. Lalesco, Sur un point de la théorie des noyaux symétrisables. M. T. Beritch, Sur la convergence et divergence des séries à termes réels et positifs. A. Buhl, Sur l'intervention de la géométrie des masses dans certains théorèmes concernant les surfaces algébriques. S. Lattès, Sur l'itération des fractions irrationnelles. De Pulligny, Sur quelques valeurs de la quadrature approchée du cercle. G. Humbert, Sur les représentations d'un entier par certaines formes quadratiques indéfinies. G. Julia, Sur les substitutions rationnelles. R. Garnier, Sur les singularités irrégulières des équations linéaires. Valiron, Sur le maximum du module des fonctions entières. De Pulligny, Quelques remarques nouvelles sur la quadrature approchée du cercle. J. Boussinesq, Équations aux dérivées partielles, pour les états ébouleux voisins de la solution Rankine-Levy, dans le cas d'un terre-plein à surface libre ondulée, mais sans pente moyenne. J. Pérès, Sur certaines développements en séries. T. Lalesco, Sur l'application des équations intégrales à la théorie des équations différentielles linéaires. M. T. Beritch, Un procédé intuitif pour la recherche des maxima et minima ordinaires. J. Andrade, Sur quelques transformations ponctuelles, et sur le cercle de similitude de deux cycles. R. Bricard, Sur le mouvement à deux paramètres autour d'un point fixe. G. Humbert, Sur les formes quadratiques indéfinies d'Hermite. G. Julia, Valeurs limites de l'intégrale de Poisson relative à la sphère, en un point de discontinuité des données. C. De la Vallée Poussin, Sur la meilleure approximation des fonctions d'une variables réelle par des expressions d'ordre donnée. J. Pérès, Quelques remarques sur certains développements en séries. A. Buhl, Sur les séries de polynomes tayloriens franchissant les domaines W.

Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 25, Heft 4–12.

I. Schur, Über die Kongruenz $x^m+y^n\equiv z^m$ (mod. p). R. Mehmke, Über die Halbierungslinien der Winkel eines Vielecks. W. Blaschke, Eine Frage über konvexe Körper. K. Boehm, Über die Entwickelung rationaler Funktionen nach Potenzen von

Polynomen. A. Korselt, Auflösung einiger Paradoxien. Über eine diophantische Aufgabe. E. Haentzschel, Über eine Aufgabe aus der Arithmetik des Diophant. A. Sassmannshausen, Zur Theorie der linearen Integrodifferentialgleichungen. K. Reinhardt, Über die kleinste Kugel, die um jede Punktmenge vom Durchmesser Eins gelegt werden kann. P. Mahlo, Über Teilmengen des Kontinuums von dessen Mächtigkeit. E. Müller, Die achsiale Inversion. E. J. Gumbel, Eine Darstellung statistischer Reihen durch Euler. H. Weyl, Strenge Begründung der Charakteristikentheorie auf zweiseitigen Flächen. A. Rosenthal u. O. Szasz, Eine Extremaleigenschaft der Kurven konstanter Breite. P. Riebensell, E. Busche. E. Haentzschel, Über die Kongruenz $2^{1092}\equiv 1$, mod. 1093^2. v. Sz. Nagy, Über die algebraische Darstellung der verknoteten und verketteten algebraischen Raumkurven. M. Bauer, Zur Bestimmung der reellen Wurzeln einer algebraischen Gleichung durch Iteration. J. Horn, Verallgemeinerte Laplacesche Integral als Lösungen linearer und nichtlinearer Differentialgleichungen. E. Lampe, Zur mechanischen Quadratur. E. Haentzschel, Theorie der Dreiecke mit rationalen Masszahlen der Seiten und der drei Seitenhalbierenden. A. Korselt, Über eine Diophantische Aufgabe. P. v. Schaewen, Bemerkungen zu den Abhandlungen des Herrn Haentzschel im 24. Bande, S. 467 ff. und im 25. Bande, S. 139 ff. E. Haentzschel, Bemerkung zu der vorstehenden Notiz des Herrn v. Schaewen. Hans Hahn, Über Fejérs Summierung der Fourierschen Reihe. H. Liebmann, Die Transformation von Variationproblemen. A. Kneser, Eine durch elliptische Funktionen darstellbare Transformationsgruppe. L. Koenigsberger, Weierstrass' erste Vorlesung über die Theorie der elliptischen Funktionen.

Journal für die reine und angewandte Mathematik, Bd. 1.7.

K. Hensel, Allgemeine Theorie der Kongruenzklassengruppen und ihrer Invarianten in algebraischen Körpern. T. H. Gronwall, Über einige Summationsmethoden und ihre Anwendung auf die Fouriersche Reihe. O. Perron, Über Systeme von linearen Differenzengleichungen erster Ordnung. A. Kneser, Transformationsgruppen und Variationsrechnung. O. Koebe, Abhandlungen zur Theorie der konformen Abbildung. III. Der allgemeine Fundamentalsatz der konformen Abbildung nebst einer Anwendung auf die konforme Abbildung der Oberfläche einer körperlichen Ecke. L. Koschmieder, Konjugierte Punkte und Enveloppen bei speziellen Variationsproblemen. O. Szász, Über die Erhaltung der Konvergenz unendlicher Kettenbrüche bei independenter Veränderlichkeit aller ihrer Elemente. F. Schottky, Problematische Punkte und die elementaren Sätzen, die zum Beweise des Picardschen Theorems dienen. R. Fueter, Die Klassenzahl zyklischer Körper vom Primzahlgrad, deren Diskriminante nur Primzahl enthält. M. Pasch, Grundfragen der Geometrie. A. Ostrowski, Über sogenannte perfekte Körper. J. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. K. Hensel, Die Verallgemeinerung des Legendreschen Symboles für allgemeine algebraische Körper. M. Pasch, Sechs Punkte einer Ebene.

Mathematische Annalen, Bd. 78, Heft 1–2.

A. Loewy, Über Matrizen- und Differentialkomplexe. R. Schauffler, Über wiederholte Funktionen. R. Koenig, Grundzüge einer Theorie der Riemannschen Funktionenpaare. A. Ostrowski, Über die Existenz einer endlichen Basis bei gewissen Funktionensystemen. A. Haar, Reihenentwicklungen nach Legendreschen Polynomen. E. Hilb, Zur Theorie der linearen funktionalen Differentialgleichungen. H. Mohrmann, Über algebraische und nichtalgebraische gewundene Kurven n-ter Ordnung vom Maximalindex. R. Baer, Über Greensche Randwertaufgaben bei der Schwingungsgleichung.

G. Hessenberg, Vektorielle Begründung der Differentialgeometrie. J. A. Schouten, Zur Klassifizierung der assoziativen Zahlensysteme.

Zeitschrift für Mathemstik und Physik, Bd. 64, Heft. 3–4.

R. v. Mises, Graphische Statik räumlicher Kräftsysteme. F. Willheim, u. A. Leon, Über das elastische Gleichgewicht von zylindrischen Ringen und die Spannungsverteilung in einem gelochten Zugstabe von endlicher Breite. A. Ritter, Spannung in einem auf mehreren Stützen gelagerten Hohlzylinder unter der Wirkung von in der Richtung der Zylinderachse gleichmässig verteilten Belastungen. H. Leitz, Berechnung der eingespannten rechteckigen Platte. F. Schilling, Neue Methoden der Ortsbestimmung eines Fremdkörpers, insbesondere eines Geschosses, im menschlichen Körper durch Röntgenaufnahmen, ein einfaches Beispiel der Photogrammetrie.

Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, Bd. 47, Heft 9–12. Bd. 48. Heft 1–5.

W. Lietzmann, Kreislehren für den mathematischen Unterricht an höheren Knabenschulen. H. Wieleitner, Zur Erfindung der analytischen Geometrie. H. Scheffler, Über Peripheriewinkel und Zentriwinkel der Ellipse. A. Lanner, Der didaktische Wert der Arnebergschen Kegelkonstruktionen. F. R. Scherrer, Die Struktur der Heronischen Dreiecke. H. Beck, Die Hessesche Normalform. H. Ruff, Das Logarithmieren von Ungleichungen. W. Brunner, Anwendung des Flächensatzes zum Nachweis der Erddrehung.

P. Luckey, Kriegsnomogramme. R. Böger, Pappus-Fünfeck-Steiner. A. Gutzmer, Die Tätigkeit des Deutchen Unterausschusses der Internationalen Mathematischen Unterrichtskommission 1908–1916. R. Ullrich, Über das Gleiten und Rollen eines Körpers entlang einer schiefen Ebene. F. Pugehl, Die Behandlung der Viereckslehre. P. Luckey, Kriegsnomogramme. A. Peter, Das stabile Schwimmen mathematischer Körper. F. Pugehl, Die Behandlung der Viereckslehre. A. Carl, Zur Zinseszinsformel. P. Zühlke, Eine analytisch-geometrische Lösung des Systems zweier allgemeiner quadratischer Gleichungen mit zwei Unbekannten durch eine kubische Gleichung. K. Giebel, Das Stangenplanimeter. P. Schwarz, Über die Beobachtung als Quelle eines Satzes der "Natürlichen Geometie". O. Herrmann, Zur Lehre von der Krümmung ebener Kurven.

Archiv der Mathematik und Physik, Bd. 26.

C. Caratheodory und H. Rademacher, Über die Eindeutigkeit im Kleinen und im Grossen stetiger Abbildungen von Gebieten. E Magin, Die stereographische Projektion der Kegelschnitte. K. Popoff, Sur la notion de l'aire d'une surface. E. Jahnke, Zur Theorie der vierdimensionalen Vektoren und Dyaden. H. Du Bois, Gewisse ebene Blattkurben und deren elektromagnetische Bedeutung L. Braude, Über die Einhüllenden gewisser Kreis- und Kugelscharen. E. Müller, Kreise als Loxodromen. F. A. Willers, Graphische Integration einiger gewöhnlichen Differentialgleichungen erster Ordnung mittels Strahlkurven. K. Knopp, Einheitliche Erzeugung und Darstellung der Kurven von Peano, Osgood und v. Koch. W. Blaschke, Über eine Ellipseneigenschaft und über gewisse konvexe Kurven. T. Wahlen, Beiträge zur Ballistik. O. Szàsz, Über arithmetische Eigenschaften gewisser unendlicher Zahlenfolgen und zugehöriger Potenzreihen. J. Horn, Analytische Lösungen von Summengleichungen Rezensionen.

Giornale di Matematiche di Battaglini, (3), Vol. 55, Set. 1917.

V. Gallico, Sulle condizioni iniziali che determinano gli integrali delle equazioni differenziali ordinarie. G. Usai, Le funzioni di Lamé dei primi 12 gradi. W. Sierpinski, Sur un problème de M. Lusin. E. Pascal, Su di un derivatore polare da servire nella radiotelegrafia. G. Scorza, Deduzione di un teorema del Prof. Capelli della teoria delle involuzioni rettilinee di specie qualunque. E Pascal, L'integrazione doppia nel campo complesso. R. Serini, L'azione hamiltoniana per le superficie di rivolu zione.

Atti della Reale Accademia dei Lincei, (5), Vol. 27, Fasc. 4–7 (1 Semestre).

G. Sannia, Sulle serie di potenze sommate col metodo di Borel generalizzato. S. L. Bianchi, Sulla integrazione dell'equazione $rt-s^2+c(p^2+q^2)^2=0$. G. Scorza, Sulle curve ellittiche singolari. A. Antoniazzi, Sopra il movimento di rotazione diurna della Terra. C. G. Loria, Fasci di quadriche rotendo e curve cartesiane. E. Bompiani, Nuovi criteri per l'isometria di due superficie o varietà. R. Serini, Euclideità dello spazio completamento vuoto nella relatività generale di Einstein. C. Minéo, Sopra un caso limite notevole di triangoli geodetici.

Atti della Reale Accademia delle Scienze de Torino, Vol. 53, Disp. 5–9, 1917–1918.

G. Mori Breda, Estrazione graduale della radice quadrata. P. Straneo, Relazioni generali fra teorie fisiche e costanti universali. C. Burali-Forti, Linea in ogni cui punto è assegnata una direzioni invariabilmento collegata al triedro principale. Lagneau Logique des propositions.

雜　錄　彙　報

歐　米　諸　大　學　ノ　課　程

北　米　合　衆　國

こ　ろ　ん　び　あ　大　學 (1918 夏學期)

まくれー J. Maclay, 數學ノ根本概念 (5), 高等代數學 (5). かすなー E. Kasner, 近世幾何學 (5), 連續群論 (5). ふぁいと W. B. Fite, 實變數ノ函數論 (5), 微分方程式論 (5).

同 (1918–1919)

ふぃすく T. S. Fiske, 函數論 (4). こーる F. N. Cole, 代數學 (4). まくれー J. Maclay 表面ノ微分幾何學 (4, 前半年間), 楕圓函數ノ應用 (3, 後半年間). すみす D. E. Smith, 數學史 (2), 數學史ニ於ケル實習 (4). けーぜる C. J. Keyser, 數學哲學 (4). かすなー E. Kasner, 彈道學 (2, 後半年間), 微分幾何學ノ「セミナリー」(2). ふぁいと W. B Fite, 微分方程式論 (3). ふぃっしゃー C. A. Fischer, 變分法 (3, 後半年間).

かーねる大學 (1918 夏學期)

かぁーゔぁー W. B. Carver, 敎師用幾何學 (3). ふるうぃつ W. A. Hurwitz, 敎師用代數學 (3). おーゑんす F. W. Owens, 射影幾何學 (3).

同 (1918–1919)

まくまほん J. McMahon, 公算論 (3), 保險學序論 (3). たんなー J. H. Tanner, 經濟數學序論 (2). すないだー V. Snyder, 畫法幾何學 (3, 第一學期間), 空間解析幾何學 (3, 第二學期間). しぁーぷ F. R. Sharpe, 流體力學 (3, 第一學期間), 彈性力學 (3, 第二學期間). かぁーゔぁー W. B. Carver, 射影幾何學 (3). れーなむ A. Ranum, 線幾何學 (3, 第二學期間). ぎれすぴー D. C. Gillespie, 微分方程式論 (3). ふるうぃつ W. A. Hurwitz, 數理物理學ニ於ケル微分方程式論 (3). くれーぐ C. F. Craig, 複素變數ノ函數論 (3). おーゑんす F. W. Owens, 高等微積分學 (3). がぁばぁ M. G. Gaba, 數學問題 (3).

じょんす, ほっぷきんす大學 (1918–1919)

もーりー F. Morley, 高等幾何學 (3, 第一學期間), 函數論 (3, 第二學期間), 一般力學及ビ流體力學 (2, 第二學期間). こぶる A. B. Coble, 對應論 (2). こーへん A. Cohen, 初等函數論 (2), 應用數學 (2, 第二學期間).

かりほるにあ大學 (1918 夏學期)

のーぶる C. A. Noble, 解析幾何學 (5), 敎師用數學 (5). すないだー V. Snyder, 微積分學 (5), 高等幾何學 (5). (以上 ろす, あんぜるすニ於テ).

れーまー D. N. Lehmer, 數論 (5). だにえる P. J. Daniell, 微分方程式論 (5), 積分方程式論 (5). べるんしゅたいん B. A. Bernstein, 複素變數函數論 (5). (以上 ばーくれーニ於テ).

しかご大學 (1918 夏學期)

むーあ E. H. Moore, 一般解析學ニ於ケル微分方程式論 (4, 前半期間), 專門敎育代數學 (5, 前半期間). ぶりす G. A. Bliss, 變分法 (4), 高次ノ平面曲線 (4). ぢっくそん L. E. Dickson, 數係數方程式ノ解法 (4, 前半期間), 行列式及對稱函數論 (4, 後半期間), 代數不變式論 (4). すろーと H. E. Slaught, 定積分 (4), 積分學 (4). らん A. C. Lunn, 單位及ビ「デメンション」(4), 電磁氣學 (4). やんぐ W. A. Young 幾何學事項 (4), 平面解析幾何學 (5). りちゃーどそん R. G. D. Richardson, 複素變數ノ函數論 (4). 微分學 (5). ろーばー W. H. Roever, 畫法幾何學 (5), 平面三角法 (5). まいあーす G. W. Myers, 中等學校(師範學校)數學敎授法 (5).

同 (1918 秋學期)

むーあ E. H. Moore, 一般解析學ニ於ケル「マトリツクス」(3). ぢっくそん L. E. Dickson, 數論 (3). うぃるちんすきー E. J. Wilczynski, 射影微分幾何學 (3). すろーと H. E. Slaught, 微分方程式論 (3). らん A. C. Lunn, 熱及ビ分子物理學 (3), 電子論 (3).

同 (1918–1919 冬學期)

むーあ E. H. Moore, 無限多變數ノ函數論 (3). ぢっくそん L. E. Dickson, 代數的數論 ぶりっす G. A. Bliss, 定積分 (3), 微分方程式論 (2). うぃるちんすきー E. J. Wilczynski, 射影微分幾何學 (3). らん A. C. Lunn, 熱力學 (3), 音響學 (3).

同 (1919 春學期)

むーあ E. H. Moore, 無限多變數ノ函數論 (3). 極限及ビ級數論 (3). ぢっくそん L. E.

Dickson. 線式代數學 (3), 立體解析學 (3). ふりっす G. A. Bliss, 線函數論 (3), 偏微分方程式論 (3, 複素變數ノ函數論 (3). らん A. C. Lunn, 幾何光學 (3).

いりのいず大學 (1918 夏學期)

けんぷなー A. J. Kempner, 高等代數學. しょー J. B. Shaw, 微分方程式論, 動徑解析. しざむ C. H. Sisam, 立體解析幾何學, 微分幾何學.

同 (1918-1919)

たうんせんど E. J. Townsend, 複素變數ノ函數論 (3), 微分方程式論及ビ高等微分學 (3), みらー G. A. Miller, 連續群論 (3, 第二學期間). 方程式論 (3, 第一學期間). りーつ H. L. Rietz, 保險數學 (3. すてびんぐす J. Stebbings, 最小自乘法 (2, 第一學期). しょー J. B. Shaw, 基礎函數 (3, 第一學期), 函數的變換論 (3, 第二學期間). しざむ C. H. Sisam, 不變式及ビ高次平面曲線論 (3), 立體解析幾何學 (3, 第二學期間). かーるみかえる R. D. Carmichael, 楕圓函數論 (3). えむひ A. Emch, 射影幾何學 (3). くらそーん A. R. Crathorne, 變分法 (3). らぃとる E. B. Lytle, 數學史 (2, 第二學期間). わーりん G. A. Wahlin, 數論 (3).

うぃすこんしん大學 (1918 夏學期)

むーあ R. E. Moore, 微分方程式論(初步) (5), 三角法及ビ立體幾何學. すきんなー E. B. Skinner, 微分幾何學 (3). 中等教育數學教授法 (5) 及代數學. まーち H. W. March, 剛體力學 (5), 公算論 (3), 微積分學 (5). しんぷそん T. M. Simpson, 三次以上ノ方程式論 (3). 積分學 (12). ゔぁんゔれっく E. B. Van Vleck, n 個變數ノ線狀置換論 (3). 初等數學ノ研究及ビ共發達史 (5).

いんぢあな大學 (1918 夏學期)

だゔぃそん Davisson, 非ユークリッド幾何學, 利息算ノ理論. ろーすろっく Rothrock, 高等積分學, 解析幾何學並ニ球面三角法. はんな Hanna, 微分方程式及微積分學. 外ニ學校教育ニ於ケル代數, 三角法及ビ解析幾何學ノ初步ノ講演.

同 (1918-1919)

だゔぃそん S. C. Davison, 微分方程式 (3), 利息算 (3), 非ユークリッド幾何學 (2). ろーすろっく D. A. Rothrock, 高等解析幾何學 (3), 方程式及行列式論 (2), 高等微積分學 (3), 數學史 (2). はんな U. S. Hanna, 解析力學 (3), 高等代數學 (2).

かんさす大學 (1918 夏學期)

あっしゅとん C. H. Ashton, 專門教育代數學, 靜力學. すつーふぁー E. B. Stouffer, 微積分學, 近世幾何學. ほいーらー J. J. Wheeler, 立體幾何學, 三角法, 解析幾何學. みっちぇる U. G. Mitchell, 初等數學史, 師範課程.

みしがん大學 (1918 夏學期)

課程：微積分學, 微分方程式論, 高等代數學, 射影幾何學, 數學史, 經濟及保險數學,「ポテンシャル」論.

講師：び｜まん Beman, ふぉ｜ど Ford, か｜ぴんすき｜ Karpinski, ぶらどしょ｜ Bradshaw, あれん Allen, ねるそん Nelson, る｜す Rouse, こ｜ Coe.

ころらど大學 (1918 夏學期)

課程：初等數學，師範課程，微積分學，微分方程式論．最小自乘法，フ｜リ｜級數．射影幾何學，微分幾何學，ガロア｜理論．定積分．

講師：らいと G. H. Light, ふぃんける B. F. Finkel, こ｜へん A. Cohen.

まさしゅ｜せっつ工藝研究所 (1918 夏學期)

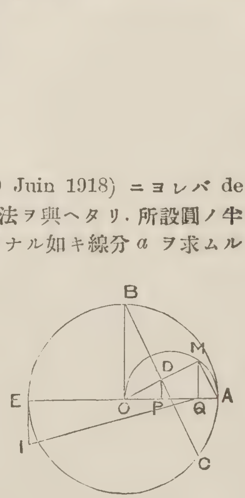
む｜あ C. L. E. Moore, 立體解析幾何學，微積分學．ぱっさの L. M. Passano, 積分學．ふぃりっぷす H. B. Phillips, 解析幾何學，初等微積分學．ぢょ｜ぢ N. R. George, 立體幾何學，初等代數學．

かりふぉるにあ大學ノ數學歷史講座

先キニ長ク Colorado College ノ教授トシテ，又數學史及物理學史ノ著者トシテ令名アリテ，我國ニ其名ノ知ラレタル Professor Florian Cajori ハ University of California ニ於ケル Professor of the History of Mathematics トシテ就任シタリ．是レ米國ニ於テ大學ノ講座トシテ數學史ノ特立セラレタル嚆矢ニシテ，歐洲ニ於ケル Strassburg 大學ガ Professor Max Simon ヲ Honorary professor of the history of mathematics トナセシ場合ノ外類例ヲ見ザル所ナリトイハル．實際數學史ノ包含ハ極メテ廣大ニシテ其ノ講義モ漸ク重要視セラレ來タリ特立ノ講座ヲ置クコト至當ナリトノ說行ハルヽニ至レリ．因ニイフ Simon 教授ハ本年 1918 一月七十四歲ノ高齡ニテ逝去セリ (T. H.)

圓積問題ノ近似解

佛國學士院ノ Comptes Rendus, t. 166, No. 23, p. 941, (10 Juin 1918) ニヨレバ de Pulligny 氏ハ所設圓ノ面積ニ近似セル正方形ヲ定ムル簡單ナル作圖法ヲ與ヘタリ．所設圓ノ半徑ヲ長サノ單位ニトレバ求ムル正方形ノ一邊ヲ求ムルコトハ $\pi \fallingdotseq a^2$ ナル如キ線分 a ヲ求ムルコトナリ．所設圓ノ中心ヲ O トシ，直徑 AE ヲ引ク．AO ヲ直徑トスル半圓周上ノ點 M ヨリ AE ニ垂線 MQ ヲ下シ，AO ノ中點 S ヲ通リ，OM ト點 D ニテ直交スル原圓ノ弦 BC ヲ引キ，D ヨリ AE ニ垂線 DP ヲ下ス．今 $OS=\frac{1}{2}$, $2PS=AQ$ ニ注目スレバ直角三角形ノ性質ニヨリテ



$$DS^2=OS.PS=\frac{1}{2}PS=\frac{1}{4}AQ.$$

又

$$BD^2=\frac{1}{4}BC^2.$$

$$OD^2=OB^2-BD^2=1-\frac{1}{4}BC^2,$$

$$OD^2 = OS^2 - DS^2 = \frac{1}{4} - \frac{1}{4}AQ.$$

故ニ

$$1 - \frac{1}{4}BC^2 = \frac{1}{4} - \frac{1}{4}AQ, \qquad \text{卽チ} \quad BC^2 = 3 + AQ.$$

從ツテ AQ ガ π ノ小數部分ノ近似値ヲ與フル如ク Q ヲ定ムレバ BC ハ求ムル正方形ノ一邊ヲ與フ. 而シテ點 Q ヲ定ムルニハ

E ニ於テ切線 EI ヲ作リ, $EI = \frac{1}{4}$ トシ, $IQ = 2 - \frac{1}{8}$ トスレバ可ナリ. 何トナレバ此場合ニハ

$$EQ = \frac{1}{8}\sqrt{221} = 1.85826 \text{ (切リ上ゲ)}$$

$$\therefore \quad AQ = 2 - EQ = 0.14174\cdots\cdots$$

故ニ

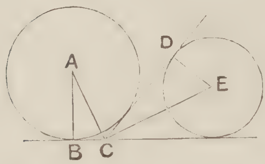
$$BC^2 = 3.14174\cdots\cdots \quad \text{ナレバナリ.}$$

和　算　ノ　一　問　題

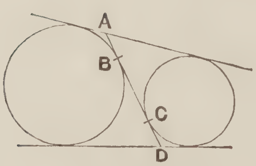
次ノ問題ハ馬場正督ノ著自問自答中ニ載セラレタルモノニシテ, 東京物理學校雜誌第 318 號(大正七年五月發行)ニモ掲ゲラレタルモノナリ. 而シテ茲ニ示シタル證明ハ仙臺市立工業學校教諭四野宮今朔治氏ノ手ニナルモノナリ.

問題. 一ツノ圓ニ外接スル二個ノ n 邊凸多角形ヲ畫キ, カクテ圓外ニ生ズル $2n$ 個ノ三角形ノ內接圓ノ半徑ニ於テ一ツオキニトリタルモノノ積ハ相等シ.

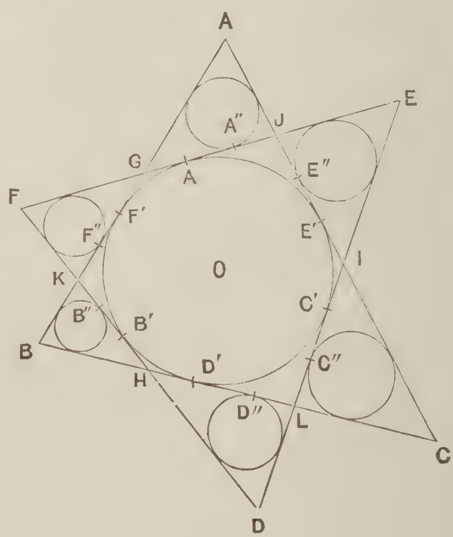
補助定理. 次ノ關係ハヨク知ラレタリ.



(1)　$AB : BC = CD : DE$



(2)　$AC = BD$



本問ノ解. 簡單ノタメ $n=3$ トス. 然レドモ其論法ハ一般ノ場合ニ通ズルコト明カナリ.

二ツノ三角形ヲ ABC, DEF トシ，邊ノ交點ヲ夫々圖ノ如ク $G, H, I; J, K, L$ トシ，$\triangle AGJ$ ノ邊 GJ ト原圓 O トノ切點ヲ A'，內接圓トノ切點ヲ A'' ニテ表シ，此內接圓ノ半徑ヲ a ニテ表ス如クス，今原圓 O ノ半徑ヲ單位ニトレバ

$$E'J:1=a:A''J.$$

$$\therefore\quad A''J=a/E'J.$$

同理ニヨリテ

$$F''G=f/A'G=f/A''J=f.\ E'J/a,$$

$$B''K=b/KF'=b/F''G=ab/f.\ E'J,$$

$$D''H=d/HB'=d/B''K=df.\ E'J/ab,$$

$$C''L=c/LD'=c/D''H=abc/df.\ E'J,$$

$$E''I=e/Ic'=e/C''L=def.\ E'J/abc.$$

然ルニ $E''I=E'J$ ナル故

$$1=def/abc,\qquad \text{卽チ}\qquad abc=def.$$

既　知　函　數

代數函數，指數函數，圓函數ヲ基礎トシテ，函數ノ函數或ハ方程式ノ根トシテ定義サルベキ總テノ函數ガ，初等數學デ往々既知函數ト呼バルヽモノヽ範圍デアルラシイ．之ヲ今少シ統一シタ言葉デ述ベルト次ノ樣ナ定義ニナル．

任意個ノ變數 $x_1, x_2, \ldots, x_n$ ト及 $e^{x_1}, e^{x_2}, \ldots e^{x_n}$ トノ間ノ一次聯立方程式デ陰伏的ニ與ヘラルヽ關係ヲ既知函數ト云フ．

例ヘバ $z=xy$ ハ次ノ一次聯立式デ與ヘラルヽ關係デアル．

$$z=e^w,\quad x=e^u,\quad y=e^v,\quad w=u+v.$$

又例ヘバ $y=\sin x$ ハ次ノ關係ト同ジデアル．

$$u=\frac{1}{2i}e^{u_1},\qquad v=\frac{1}{2i}e^{v_1},\qquad u_1=ix,\qquad v_1=-ix,\qquad y=u-v.$$

其他如何ニ複雜ナ既知函數デモ，適當ニ分解スレバ，上ノ定義ニ含マルヽ事ガ證明サレル．

此定義ヲ基礎トシテ楕圓函數ハ既知函數ニアラズト云フ事ガ證明サレナイモノデ有ラウカ (S. K.)

全　微　分　ノ　條　件

連續函數 P, Q ニ關スル微分式

$$P(x,y)dx+Q(x,y)dy$$

ガ或函數ノ全微分 (total differential) デアル爲メノ必要且十分ナル條件ハ

$$\frac{\partial}{\partial y}P(x,y)=\frac{\partial}{\partial x}Q(x,y)$$

ナリト云フノガ普通デアル．シカシ之ハ P 及 Q ノ微係數ガ存在シテ連續デアルモノト假定シテノ上ノ事デアル．

P 及 Q ノ可微分性ヲ絕對ニ假定シナイデ言ヘバ，條件ハ，任意ノ x_1, x_2 ニ對シテ

$$\frac{\partial}{\partial y}\int_{x_1}^{x_2}P(x,y)dx=Q(x_2,y)-Q(x_1,y)$$

ガ成立スルト云フ事ニナル．勿論之ノ代ハリニ

$$\frac{\partial}{\partial x}\int_{y_1}^{y_2} Q(x,y)dy = P(x,y_2) - P(x,y_1)$$

ト云フ條件ヲ取ツテモ宜シイ．

之ハ微積分ノ好イ演習問題デアラウト思フ (S. K)

代數函數ノ解析接續

代數方程式 $f(x,y)=0$ カラ定義サルヽ代數函數 $y(x)$ ガ n 値函數デアルトシテ，x ガ x' カラ x'' マデ複素數平面上ノ或道 L ニ沿フテ進ム時，y ノ n 個ノ値ハ $y_1', y_2', \cdots, y_n'$ カラ $y_1'', y_2'', \cdots, y_n''$ マデ變ハルモノトスル．此時此二組ノ値ガ如何ナル置換ヲ受ケルカ卽第一群ノ何レノ値ガ第二群ノ何レノ値ニ變ハルカト云フ事ハ，L ノ形ニ關係シテ定マル．之ヲ定メルニハ一般ノ函數論的方法デ言ヘバ，L ニ沿フテ所謂 analytic continuation ヲ行ハネバナラヌ．所ガ代數函數ノ場合ニハ別ニ簡單ナ方法ガアル．通常ノ函數論ノ書物ニ見當ラヌ樣デアルカラ玆ニ述ベル．

先ヅ $x=\xi_1+i\xi_2,\quad y=\eta_1+i\eta_2$

ト置ケバ，原ノ方程式ガ分レテ二ツノ實方程式

$$A(\xi_1,\xi_2,\eta_1,\eta_2)=0,\qquad B(\xi_1,\xi_2,\eta_1,\eta_2)=0$$

トナル．之カラ η_2 ヲ追出セバ $C(\xi_1,\xi_2,\eta_1)=0$ ヲ得ル．η_1 ニ關シテ之ノ判別式 discriminant ヲ取レバ ξ_1, ξ_2 ノ代數方程式 $D(\xi_1,\xi_2)=0$ ヲ得ル．之ハ x 平面上ノ一ツノ曲線 K ヲ表ス．K ト L トノ交點ヲ L ノ上ニ順々ニ求メテ $x_1, x_2, \cdots, x_m$ トスル．

x ガ x' カラ x'' マデ L 上ヲ動ク間ニ y ノ n 個ノ値ノ實數部ガ相等シクナリ得ル事ハ $x_1, x_2, \cdots, x_m$ ニ於テノミデアル．故ニ $y_1', y_2', \cdots, y_n'$ ト x_1 ノ直グ前ニ於ケル y ノ n 個ノ値 $y_1, y_2, \cdots, y_n$ トノ間ノ continuation ノ模樣ハ實數部ノ大サノ順序ガ變ハラナイト云フ事カラ直グ分カル．又 x_1 ノ直グ前ト直グ後トニ於ケル y ノ n 個ノ値ノ移リ行ク模樣ハ虛數部ノ大サノ順序デ直グ分カル．次ニ又 x_1 ノ直グ後ト x_2 ノ直グ前トニ於ケル y ノ n 個ノ値ノ間ノ continuation ハ實數部ノ大サノ順序ガ不變ナル事カラ直グ分カル．追テ斯樣ナ吟味ヲ重ネテ行ケバ，結局 $x_1, x_2, \cdots, x_m$ ニ於テ夫々 y ノ枝函數ノ展開ヲ第二項マデ求メテ置ケバ，x' ト x'' トノ間ニ於ケル y ノ値ノ continuation ガ知リ得ラルヽノデアル (S. K.)

Frobenius 教授ヲ悼ム

伯林大學教授 Georg Frobenius 氏ハ昨年 (1917) 八月六十八歲ヲ以テ遂ニ逝カレタ．

彼ハ 1849 ニ生レタ．最初 Göttnigen 大學デ Stern, W. Weber ニ學ビ，後 Berlin 大學ニ移ツテ Weierstrass, Kummer, Kronecker ニ學ビ，1870 同大學カラ，或ル整函數ヲ項トスル級數ニ由ル解析函數ノ展開ニ關スル論文ニヨツテ Dr. ノ學位ヲ得タ．1876 頃カラ 1892 マデハ Zürich ノ高等工業學校ニ教鞭ヲ執リ，1893 伯林大學ニ移リ，同時ニ Schwarz ト前後シテ伯林學士院ニ入ツタ，之レ實ニ Kronecker ノ後繼者トシテデアツタ．

近來ニナツテ獨逸數學界ノ耆宿モ追々ト凋落シテ來タ，ソレデモ Frobenius ト同時ニ伯林大學デ Dr. ニナツタ人ニ Netto, Kiepert, J. König, Rosanes ガアリ，彼以前ニ Dr. ニナツタ人ニハ Schwarz (1864), Mertens (1864), Lampe (1864) Pasch (1865), G. Cantor (1867), Klein (1868) ガアル．

Frobenius ハ純然タル代數學者デアツタ．彼ガ年々伯林大學デ講ズル題目モ殆ンド數論，代數學ノ範圍ニ限ラレ，彼ガ公ニシタ百有餘ノ論文中代數學的色彩ヲ帶ビナイモノハ幾何モナイ．

彼ガ伯林ニ歸來スルマデハ微分方程式論ヤ楕圓函數，しーたー函數ニツイテ多クノ論文ヲ出シタガ，ソレノ大部分ハ代數學的方面ガ主デアツタ．此ノ時代ノ論文ハ殆ンド Crelle Journal デ公ニサレタ．伯林學士院ニ入ツテカラハ彼ハ學士院ノ報告ヲ殆ンド唯一ノ發表所トシテオツタ．(Math. Ann. 70, 1911 ニハ奇ラシク彼ノ一論文ガ載ツテオル.) カヽル態度ノ人ハ餘リ多クヲ求メラレナイ．彼ハ確ニ一種ノ風格ヲ備ヘタ學者デアツタ．

彼ハ非常ニ獨創的ナ人トハイヒ得ラレナイデアラウ．然シ彼ノ論文ヲ讀ムト彼ナラデハト思ハレル獨特ノ技巧ニ魅セラレル．彼逝キテ伯林大學ニ於ケル Dirichlet, Kummer, Kronecker, Frobenius ノ傳統ハ果シテ誰ニ由リテ繼ガレルデアラウカ．

余ハ下ニ彼ノナシタ研究ノ主要ナルモノヲ列擧シテ亡キ彼ヲ追憶シヤウト思フ．

行列式トまとりっくす論デ最モ根本的ノ概念ノ一ハ Rank ノソレデアラウ．此概念ヲ導入シタノハ Frobenius 其人デアツタ．彼ハ "Pfaff ノ問題ニ就テ" (Crelle J., 84, 1877) ニ於テ此ノ概念ノ意義ノ大ナルコトヲ示シ，"完全齊一微分方程式ニ就テ" (Crelle J. 86, 1879) ニ於テ始メテ Rang ナル名稱ヲ與ヘテオル．

Weierstrass ノ樹立シタ elementary divisor ノ理論ノ一ツノ罅漏ヲ充シテ之ヲシテ完全ナラシメタノハ又 Frobenius デアツタ (Crelle J. 86, 1879-1880).

彼ハ双線狀形式ノ線狀變換及ビまとりっくすノ理論ヲ記號的ニ取扱フコトニ由リテ此レ等ノ理論ニ統一ト簡明トヲ與ヘタ (Crelle J., 84, 1878). 對稱的双線狀形式ノ合同變換ニ關スル論文 (Berliner Sitzungsber. 1896) モ又重要ナルモノデアル．

彼ノ此方面ノ研究ハ頗ル豐富デアツテ，或ハ之ヲ多元しーたー函數ノ變換ニ應用シ，或ハ之ヲ Pfaff ノ微分方程式ヤ完全微分方程式ニ應用シテ成功シテヰル．

伯林時代ニナツテカラハ群論ノ方面ノ研究ガ其基調ヲナシタ．Sylow ノ定理ノ新證明ヤ擴張ヲ以テ始マリ，群ヲ二重ノ Modul ニヨリテ分解展開スル思想ヲ導入シテ soluble group ノ爲メノ十分條件ノ研究ニ於テ功果ヲアゲタ．シカシ此ノ方面デ最モ大ナル研究ハ有限群ヲ一次ノ置換群ヲ以テ表ス問題ヲ Gruppencharaktere 及ビ Gruppendeterminante ナル概念ノ導入ニヨリテ論ジ出シタ點デアラウ (Berliner Sitzungsber. 1896 以降) 此ノ Charaktere ノ考ハ已ニ Dirichlet, Dedekind 等ニアルガ之ヲ一般ノ有限群ニ擴張シテ其意義ヲ徹底的ニ闡明スルヲ得タノハ彼ノ功績デアル．

彼ノ初期ノ研究ハ微分方程式ノ論デアツタ，其内特筆スベキハ代數方程式ニ於ケル既約ナル概念ヲ微分方程式ノ上ニ擴張シ之ニヨリテ代數方程式論ニ於ケル種々ノ定理ノ對應者ヲ微分方程式論ニ見出シタ點デアル．彼ハ之ヨリシテ既約ナラザル線狀微分方程式ノ分解ヲ論ジ特ニ之ヲ multiplicator ノ理論ト adjoint differential equations ノ理論ニ應用シテ興味アル結果ヲ收メテオル (Crelle J., 76, 1873; 77, 1874; 85, 1878) determinate singular point ニ於ケル微分方程式ノ解ヲ級數デ表ス簡單ナル方法 (Crelle J., 76, 1873) モ注意スルニ足ル．

冪函數ニ關スル Abel ノ定理ヲ擴張シテ summable series ノ研究ノ發展ヲウナガシタ (Crelle J, 89, 1880) コトモ見逃シテハナラナイ．

彼ノ晩年ハ特種ノ研究ヲ提唱シナカツタ．シカシ双線狀及二次形式ノ一次變換ヤ數論ニ關スル少壯學者ノ研究ガ一度彼ノ眼ニ觸ルルト彼ノ豐富ナル學殖ト緻密ナル頭腦トハヨリ簡單ナル方途ヤ隱レタル關係ヲ決シテ見逃サナカツタカノ感ガアル．例ヘバ Fermat 大定理ニ關スル Kummer 條件ヲ簡明ナル條件ヲ以テ置換シタル Wieferich ノ論文ニ對シテハ彼ハ直チニヨリ簡單ナル證明ヲ提供シ (Crelle J, 137, 1910; Berliner Sitzungsber. 1910) Stridsberg, Carathéodory, Bieberbach 等ノ代數學的方面ノ論文ニ對シテハ直チニ彼ナラデハ氣ノ付カナイ微妙ナ點ニ注意ヲ拂フテオル (Berliner Sitzungsber. 1911, 1912). (M. F.)

仙臺ノ理科大學數學科教師ノ擔任課目

本學年 1918-1919 ニ於ケル擔任次ノ如シ．アラビア數字ハ毎週ノ時間數ニシテ殊ニ明記ナキモノノ期間ハ皆一學年間ナリ．

林教授，微積分學 (4)，同演習 (8)，實用數學(楕圓函數應用) (2)，數學研究 (3). 藤原教授，實變數ノ函數論 (2)，同演習(2, 隔週)，代數學 (2)，同演習 (2)，座標幾何學 (九月ヨリ十二月マデ 3，一月ヨリ六月マデ 2)，同演習 (3). 窪田教授，複素變數ノ函數論 (2)，同演習 (3)，綜合幾何學 (2)，同演習 (2, 隔週)，特選題目 (線幾何學) (2). 小島助教授，代數解析 (2)，特選題目 (級數ノ總和性) (第一學期) (2). 柴山講師，微分方程式論 (3).

墺國俘虜ノ數學的著述

大正七年八月九日ノ時事新報ニヨレバ，似之島俘虜收容所俘虜墺國砲兵大尉ふぉん・もらいべっくハ他ノ三名ノ俘虜將校ト共ニ逃走ヲ企テシモ逮捕セラレシガ，同人ハ數學ニ造詣深ク曩ニ衛兵侮辱罪ニテ廣島監獄ニ禁錮收監中其ノ得意トスル數學ヲ研究シ出監後モ益々之レガ研究ニ耽リツヽアリシガ，此程二十册ノ高等數學書ヲ編纂シ世界ニ發表スル意思ニテ俘虜情報局ノ手ヲ經テ今回東京帝國大學へ寄贈スルコトヽシタリト.

大戰ノ影響

American Mathematical Society ノ第 25 Summer Meeting ハ Hanover, N. H. ニ於テ本年 9 月 4, 5, 6 ノ三日間開カレタルガ其 Preliminery notice ニヨリテ見レバ This meeting is to be devoted to the *mathematics of warfare*. It is hoped to secure as speakers men of prominence now engaged in the government service トアリ. 開會後ノ委細ニツキテハ更ニ記載ノ機アルベシ.

Prof. G. A. Miller of Illinois ノ來翰ニヨレバ Our graduate work is suffering very much on account of the war and things are getting worse rapidly. Nearly all of our best young men are going to the war and the young women seldom are good mathematicians. Hence the mathematical departments suffer especially at the present time. If we can only win, I think the sacrifice will be justified. トアリ (T. H.)

中等學校數學科教員協議會

東京高等師範學校内ノ中等教育研究會ニ於テハ本年末ヲ期シ全國師範學校中學校高等女學校數學科教員協議會ヲ催ホスノ企テアリ. 其要項竝ニ文部省諮問題，協議題及談話題ハ次ノ如ク，更ニ數學教育ニ關係アル諸氏ノ講演モアリト.

(要項)

會期　大正七年十二月二十四日ヨリ二十九日ニ至ル五日間.

會場　東京高等師範學校

事業　協議，講演，報告，談話，參觀.

會費　本協議會ニ關スル雜費トシテ御來會ノ節約金壹圓ヲ申受クルコト.

申込　御來會者官職氏名ヲ來ル十月三十一日迄ニ御申込ノコト.

(備考) 日程其他ハ十一月中御來會者ニ對シ通知ス.

(文部省諮問題)

師範學校中學校及高等女學校ノ目的ヨリ觀テ其ノ數學教授上改善ヲ要スベキ點及之ガ方案如何.

(協議題)

1. 國民ノ數學的思想ヲ一層發展增進スル爲ニ特ニ改善施設ヲ要スル事項如何.

2. 師範學校中學校及高等女學校ノ數學科ニ於テ函數及ぐらふニ關スル事項ヲ教授スル時期及程度如何.
3. 師範學校中學校及高等女學校ノ幾何教授ニ就テ幾何學入門ヲ課シ其他此ノ教授ニ於テ實驗實測ヲ加味スル方案如何.
4. 師範學校中學校及高等女學校ノ數學科ニ於テ各分科ノ連絡上特ニ注意スベキ諸點如何.
5. 師範學校中學校及高等女學校ノ數學科ニ於テ各分科ノ適當ナル配當及之レニ要スル適當ナル教授時數如何.
6. 師範學校中學校及高等女學校ノ數學教授上必要ナル設備如何.
7. 師範學校中學校及高等女學校ノ數學科ニ於テ珠算ヲ一層廣ク利用セシムルノ可否如何.

(談話題)

1. 師範學校中學校及高等女學校ノ數學科ニ於テ解析幾何學, 微積分學及力學ニ關スル事項ヲ加味シテハ如何.
2. 師範學校中學校及高等女學校ニ於テ計算ニ熟達セシムルニ適當ナル方法如何.
3. 師範學校中學校及高等女學校ニ於テ數學科ノ優等生及劣等生ノ取扱方法如何.
4. 師範學校及高等女學校ニ於テ數學科練習問題ノ取扱方法如何.
5. 師範學校中學校及高等女學校ニ於ケル數學科成績考查方法如何.

(備考) 談話題ニツキテハ單ニ意見ノ交換ニ止メ別ニ對案ノ決議ヲナサザルモノトス.

二三雜誌中ノ注目スベキ論説記事

東京物理學校雜誌, 大正 7 年 7, 8, 9, 10 月號

數理雜俎 (5, 6)		柳原吉次氏
近世幾何學(坐標ニツキ)		金子長太郎氏
てーろる圓ト關聯セル新定理		有賀牟之丞氏
$\log\varphi(t, a, n) = \log t + a\varphi^n(t, a, n)$ ナル函數		森吉太郎氏

東洋學藝雜誌, 大正 7 年 7, 8, 9, 10 月號

Baron Dairoku Kikuchi.		
序文無用ノ說, 附, 菊池博士遺著普通幾何學大要序文	理學博士	藤澤利喜太郎氏

保險雜誌, 大正 7 年 6, 7, 8, 9 月號

生命保險技術ノ發展的徑路		木村伊助氏
年金ノ利率ヲ求ムル方法ニ就テ誤ヲ正ス	理學士	竹下清松氏
邦人ノ國民死亡率ト經驗死亡率トノ推移ノ趨勢ニ就テ	理學士	角尾猛次郎氏
生命保險契約ノ效力延長ニ就テ		門脇政治氏

諸學者ノ消息

北米合衆國はーばーど大學助教授くーりっぢ氏 J. L. Coolidge ハ同大學教授トナレリ.

瑞西べるん大學教授べんてり氏 A. Benteli ハ 1917 年 11 月 10 日 70 歲ニテ逝去シ同大學教授おっと氏 E. Ott ハ 1917 年 11 月 17 日 70 歲ニテ逝去セリ.

北米合衆國最初ノ變分學書 (1881) ノ著者タルかーる氏 Lewis B. Carll ハ 1918 年 3 月 12 日 74 歲ニテ逝去セリ.

希臘あぜん大學教授すてふあの氏 Cyparissos Stéphanos ハ逝去セリ.

伊太利かすてるぬおゔお1氏 G. Castelnuovo ハりんちえい學士院會員ニ選擧セラレタリ. 佛蘭西巴里大學ノぼれる É. Borel, ぐるさ1 E. Goursat 及あだまーる J. Hadamard ノ諸教授及英國まんちえすた1大學教授らむ氏 H. Lamb ハ同學士院外國委員ニ選擧セラレタリ.

伊太利とりの學士院ハばゔぃあ大學教授べるづぉらり L. Berzolari なぽり大學教授まるころんご R. Marcolongo, ぼろにあ大學ノぴんけるれ L. Pincherle, ぱどあ大學ノりっち G. Ricci 及ぜうえり F. Severi, ぴざ大學ノあるべんが G. Albenga, ころめつち G. Colometti 及ビまっじ G. A. Maggi ろ1ま大學ノれいな V. Reina 等ノ諸氏ヲ通信會員ニ選擧セリ.

伊太利ぼろにあ大學ニ於テべるじぇりお Dr. A. Vergerio ハ同大學解析學ノ講師トナレリ.

伊太利とりの大學ニ於テとりあっち Dr. E. G. Togliatti ハ同大學射影幾何學及畫法幾何學ノ講師トナレリ.

北米合衆國かりふおるにあ大學ころらど College ノ教授かじおり氏 F. Cajori ハ數學史ノ教授ヲナスコトヽナリ. れ1ま1氏 D. N. Lehmer ハ同大學教授トナレリ.

北米合衆國 Albany ノ New York State College ノどぽると氏 J. V. De Porte ハ同大學助教授トナレリ.

伊太利ぜのあ大學れ1ゔぃ氏 Levi ハ 1917 年 10 月 28 日 34 歳ニテ戰死セリ. 伊太利科學會ニテハ氏ニ 1912 年度金賞牌ヲ授與シタリ.

伊太利ぱゞゔぃあ大學ノゔぃてるび氏 A. Viterbi ハ 1917 年 11 月 8 日 44 歳ニテ戰死セリ.

北米合衆國ころんびあ大學教授ほうくす氏 H. E. Hawkes ハけっぺる氏 F. P. Keppel ノ後ヲ繼ギテころんびあ College ノ學長トナレリ. Connecticut College ノれいぶ氏 D. D. Leib ハ數學科教授トナレリ.

しかご大學ノれ1ん氏 E. S. Lane ハ Rice Institute ノ數學教師トナレリ.

東北帝國大學理科大學助教授理學博士掛谷宗一氏ハ數學研究ノ爲滿一ケ年間米國ヘ留學ヲ命ゼラレ, 理學士小島鐵藏氏ソノ後任トシテ大正七年九月二十一日助教授ニ任ゼラレタリ又掛谷氏ハ大正七年十月十三日米國 Boston ニ向ヒテ出發セラレタリ.

東北帝國大學理科大學講師文學士田邊元氏ハ同大學創立後間モナク就任セラレ科學概論及獨逸語ヲ教授セラレツヽアルカ今般「數理哲學研究」ト題スル論文ヲ提出シ京都帝國大學文科大學教授會ノ審査ヲ經テ大正七年八月八日文學博士ノ學位ヲ授與セラレタリ. 同氏ノ數學ノ基礎ニ關スル論文ハ時時本誌ニ掲載スル所アリタリ.

大正七年九月十九日統計學者吳文聰氏逝ケリ. 同氏ハ同學輸入ノ率先者法學博士杉享二氏ニ師事シ, 更ニ其ノ業ヲ柳澤保惠伯ニ傳ヘタリト聞ク. 皆内閣統計局ノ事業ノ中樞者ナルベシ

東京數學物理學會ノ前身タル東京數學會社ノ設立ニ斡旋セラレタル同會會員眞野肇氏ハ大正七年八月十九日逝去セラレタリ.

第三高等學校教授理學士奥山賢氏ハ大正七年十月四日本年間京都帝國大學理科大學講師ヲ囑託セラレタリ.

佛國巴里市ノ數學書籍出版書肆トシテ有名ナル Gauthier-Villars et fils ノ主人 Albert Gauthier-Villars 氏ハ此程死去セルガ其生前ノ學術界ニ對スル功勞ノ爲ニ L'Académie des sciences ノ終身書記官 Picard 氏ハ哀悼ノ辭ヲ述ベタルコト Comptes Rendus, t. 167, No. 3 (16 Juillet 1918) ニ見ユ. 外國ノ書店主ノ尊重セラルヽコト斯ノ如シ. 因ニ同店主ハ Ancien élève de l'École Polytechnique (promotion 1881) トシテ有識者ナリ.

東北數學雜誌

第拾四巻　第壹,貳號

THE TÔHOKU MATHEMATICAL JOURNAL

Edited by

T. Hayashi

with the cooperation of

M. Fujiwara, *J. Ishiwara,*
T. Kubota, *S. Kakeya.*

Vol. 14. Nos. 1, 2.

(One volume consists of four numbers.)

AUGUST, 1918

THE TÔHOKU IMPERIAL UNIVERSITY,

SENDAI, JAPAN.

The Editor of the Journal, T. HAYASHI, College of Science, Tôhoku Imperial University, Sendai, Japan, accepts contributions from any person.

Contributions should be written legibly in English, French, German, Italian or Japanese and diagrams should be given in separate slips and in proper sizes. The author has the sole and entire scientific responsibility for his work. Every author is entitled to receive gratis 30 separate copies of his memoir; and for more copies to pay actual expenses.

All communications intended for the Journal should be addressed to the Editor.

Subscriptions to the Journal and orders for back numbers should be addressed directly to the Editor T. HAYASHI, or to the bookseller Y. ÔKURA, No. 19, Tôri-itchôme, Nihonbashi, Tôkyô, Japan, or MARUZEN COMPANY, LTD., Branch Office, Kokubunchô, Sendai, Japan.

Price per volume (consisting of four numbers) payable in advance: 3 yen = 6 shillings = 6 Mark = 7,50 francs = 1.50 dollars. Postage inclusive.

大正七年八月五日印刷
大正七年八月八日發行

（定價一卷（四號分）金參圓
一號金七拾五錢但郵稅共）
（本册ニ限リ複號ニ付金壹圓五拾錢）

編輯兼發行者　東北帝國大學

印刷者　東京市神田區美土代町二丁目一番地　島連太郎

印刷所　東京市神田區美土代町二丁目一番地　三秀舍

賣捌所　東京市日本橋區通一丁目十九番地　大倉書店
振替貯金口座東京二三八番

賣捌所　仙臺市國分町百六十八番地　丸善株式會社仙臺支店
振替貯金口座東京七四番

（本誌監輯者東北帝國大學理科大學教授林鶴一ニ於テ寄稿ヲ受ケ幷ニ購入ノ註文ヲ取次ベシ）

CONTENTS.